DOCUMENT RESUME

ED 411 129		SE 057 055

AUTHOR	Borbas, Andrea, Ed.
TITLE	Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education (12th, Veszprem, Hungary, July 20-25, 1988), Volume 2.
INSTITUTION	International Group for the Psychology of Mathematics Education.
PUB DATE	1988-07-00
NOTE	356p.; For volume 1, see SE 057 054.
PUB TYPE	Collected Works - Proceedings (021)
LANGUAGE	English, French
EDRS PRICE	MF01/PC15 Plus Postage.
DESCRIPTORS	Educational Research; Elementary Secondary Education; Foreign Countries; Higher Education; *Mathematics Education

ABSTRACT

The proceedings for the annual conference of the International Group for the Psychology of Mathematics Education (PME) include the following papers: "Intervention in a Mathematics Course at the College Level" (L. Gattuso & R. Lacasse); "The Education of Talented Children" (F. Genzwein); "The Development of a Model for Competence in Mathematical Problem Solving Based on Systems of Cognitive Representation" (G.A. Goldin); "Pilot Work on Secondary Level" (M. Halmos); "On the Textual and the Semantic Structure of Mapping Rule and Multiplicative Compare Problems" (G. Harel, T. Post, & M. Behr); "Forging the Link between Practical and Formal Mathematics" (K.M. Hart & A. Sinkinson); "The Kindergartners' Understanding of the Notion of Rank" (N. Herscovics & J.D. Bergeron); "Initial Research into the Understanding of Percentages" (R. Hershkowitz & T. Halevi); "Structuring and Destructuring a Solution: An Example of Problem Solving Work with the Computer" (J. Hillel, J.L. Gurtner, & C. Kieran); "Metacognition: The Role of the 'Inner Teacher'" (I. Hirayashi & K. Shigematsu); "Formalizing Intuitive Descriptions in a Parallelogram LOGO Microworld" (C. Hoyles & R. Noss); "One Mathematics Teacher" (B. Jaworski); "Learning the Structure of Algebraic Expressions and Equations" (C. Kieran); "The Influence of Teaching on Children's Strategies for Solving Proportional and Inversely Proportional Word Problems" (W. Kurth); "Constructing Bridges from Intrinsic to Cartesian Geometry" (C. Kynigos); "Concepts in Secondary Mathematics in Botswana" (H. Lea); "A Developmental Model of a First Level Competency in Procedural Thinking in LOGO: 'May Be We're Not Expert, But We're Competent'" (T. Lemerise); "The Naive Concept of Sets in Elementary Teachers" (L. Linchevski & S. Vinner); Concrete Introduction to Programming Languages and Observation of Piagetian Stages" (F. Lowenthal); "Cognitive and Metacognitive Shifts" (J.H. Mason & P.J. Davis); "Learning Mathematics Cooperatively with CAI" (Z. Mevarech); "Mathematical Pattern-Finding Elementary School, Focus on Pupils' Strategies and Difficulties in Problem-Solving" (N. Nohda); "The Construction of an Algebraic Concept Through Conflict" (A. Olivier); "Gender and Mathematics: The Prediction of Choice and Achievement" (W. Otten & H. Kuyper); "Teaching and Learning Methods for Problem Solving: Some Theoretical Issues and Psychological Hypotheses" (J. Rogalski & A. Robert); "Student-Sensitive Teaching at the Tertiary Level: A Case Study" (P. Rogers); "Strategy Choice in Solving Additions: Memory of Understanding of Numerical Relations" (A.D. Schliemann); "Representation of Functions and Analogies" (B. Schwarz & M. Bruckheimer); "Operational vs. Structural Method of Teaching

Mathematics - Case Study" (A. Sfard); "Epistemological Remarks on Functions" (A. Sierpinska); "Formative Evaluation of a Constructivist Mathematics Teacher Inservice Program" (M.A. Simon); "Construction and Reconstruction: The Reflective Practice in Mathematics Education" (B. Southwell); "Graphical Lesson Patterns and the Process of Knowledge Development in the Mathematics Classroom" (H. Steinbring & R. Bromme); "Longer Term Conceptual Benefits from Using a Computer in Algebra Teaching" (M. Thomas & D. Tall); "The Role of Audiovisuals in Mathematics Teaching" (K. Tompa); "Specifying the Multiplier Effect on Children's Solutions of Simple Multiplication Word Problems" (L. Verschaffel, E. De Corte, & V. Van Coillie); "Is There Any Relation between Division and Multiplication?: Elementary Teachers' Ideas about Division" (S. Vinner & L. Linchevski); "The Influence of Socialization and Emotional Factors on Mathematics Achievement and Participation" (D. Visser); "Metacognition and Elementary School Mathematics" (M.A. Wolters); "The Development of the Counting Scheme of a Five Year Old Child: From Figurative to Operational" (B. Wright); "Say It's Perfect, Then Pray It's Perfect: The Early Stages of Learning about LOGO Angle" (V. Zack); and "Substitutions Leading to Reasoning" (N. Zehavi). Includes a list of author addresses. (MKR)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

PERMISSION TO REPRODUCE AND
DISSEMINATE THIS MATERIAL
HAS BEEN GRANTED BY
J. van Dormolen

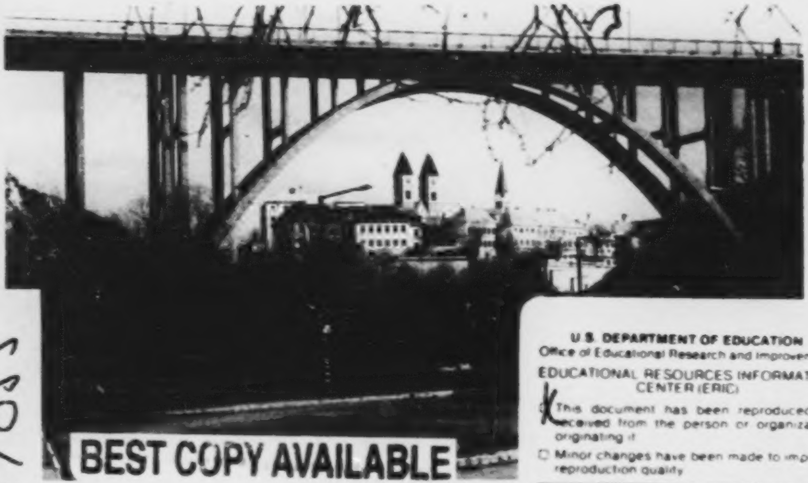
TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)

TWELFTH ANNUAL CONFERENCE
OF THE
INTERNATIONAL GROUP FOR THE
PSYCHOLOGY OF
MATHEMATICS EDUCATION

ED 411 129

PME XII.
HUNGARY 1988
20 - 25 July

PROCEEDINGS



U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)

☒ This document has been reproduced as received from the person or organization originating it

☐ Minor changes have been made to improve reproduction quality

● Points of view or opinions stated in this document do not necessarily represent official OERI position or policy

SE057055

Volume II

PROCEEDINGS

of the

12th International Conference

PSYCHOLOGY OF
MATHEMATICS
EDUCATION

Veszprém, Hungary

20 - 25 July 1988

PME 12

ISSN

ISBN

Copyright C 1988 the authors

Edited by: Andrea Borbás

Editorial board: András Ambrus
Katalin Bognár
Ferenc Genzwein
László Mérő
János Surányi

Translations: Andrea Borbás
Maria Dax

Responsible publisher: Ferenc Genzwein
general director of OOK .

Printed by OOK Printing House Veszprém
Responsible: J. Bernáth

PREFACE

The 12th annual conference of the PME is the first meeting in the history of the International Group for the Psychology of Mathematical Education held in an East-European socialist country. The conference takes place in the old episcopal city Veszprém, from July 20th to July 25th, 1988.

There are a number of different ways in which participants at the conference may make a contribution: research reports, poster displays, working groups /initiated in 1984/ and discussion groups /initiated in 1986/. One session is devoted to the preparation for the ICME-6 presentations of the PME. An innovation at this conference is that following each group of papers of similar topics a summary session will be held to discuss and evaluate the achievements in the given territory. The discussion sessions will be held in the following topics:

1. Algebra
2. Rational numbers
3. Early numbers
4. Metacognition
5. Teachers' beliefs
6. Problem solving
7. Computer environments
8. Social factors

We would like to thank Thomas A. Romberg, Claude Comiti, Kathleen Hart, Richard Lesh, Tommy Dreyfus and Colette Laborde for volunteering to chair and introduce these evaluation sessions.

87 research papers have been submitted to the conference. All of them have been evaluated by at least two reviewers and the final decision on the acceptance of the papers has been done at a session of the International Program Committee in Budapest, based on the reports of the reviewers. The members of the International Committee of the PME and the International Program Committee have served as reviewers for the submitted papers.

The order in which the research papers appear in these two volumes is alphabetic /according to the first author of the paper/ except for the invited plenary papers that are taken first. Therefore the order of the papers in the volumes does not necessarily reflect the order of presentation within the meeting itself. Any particular paper can be located by consulting either the table of contents at the beginning or the alphabetical list of contributors at the end. We would like to thank the International Program Committee, the Local Organizing Committee and the reviewers fo their assistance in the preparation of this conference.

International Program Committee:

Chairman: János Surányi /Hungary/
Secretary: László Mérő /Hungary/
Members: András Ambrus /Hungary/
Katalin Bognár /Hungary/
Joop van Dormolen /The Netherlands/
Willibald Dörfler /Austria/
Tommy Dreyfus /Israel/

Local Organizing Committee:

President: Ferenc Genzwein - general director of OOK /Hungary/
Secretaries: Andrea Borbás and Maria Dax /Hungary/

HISTORY AND AIMS OF THE PME GROUP

At eh Second International Congress on Mathematical Education /ICME 2, Exeter, 1972/ Professor E. Fischbein of Tel Aviv University, Israel, instituted a working group bringing together people working in the area of the psychology of mathematics education. At ICME 3 /Karlsruhe, 1976/ this group became one of the two groups affiliated to the International Commission for Mathematical Instruction /ICMI/.

According to its Constitution the major goals of the group are:

1./ to promote international contacts and the exchange of scientific information in the psychology of mathematical education,

2./ to promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers,

3./ to further a deeper and more correct understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

MEMBERSHIP

1./ Membership is open to persons involved in active research in furtherance of the Group's aims, of professionally interested in the results of such research.

2./ Membership is on an annual basis and depends on payment of the subscription for the current year /January to December/

3./ The subscription can be paid together with the conference fee.

The present officers of the group are as follows:

President: Pearla Nesher /Israel/
Vice-President: Willibald Dörfler /Austria/
Secretary: Joop van Dormolen /The Netherlands/
Treasurer: Carolyn Kieran /Canada/

TABLE OF CONTENTS

TABLE OF CONTENTS

VOLUME I Page

I. PLENARY PAPERS

1. Street mathematics and school mathematics
Terezinha Nunes Carraher /Brasil/ 1

2. A look at the affective side of mathematics learning in Hungarian Secondary schools
Sándor Klein, Gustav M. Habermann /Hungary/ 24

3. Beyond constructivism /Learning mathematics at school/
Perla Nesher /Israel/ 54

4. Reconstructive Learning
Leen Streefland /The Netherlands/ 75

II. RESEARCH PAPERS

1. Perceptions of teachers'questioning styles
Janet Ainley 92

2. Teacher change as a result of counselling
Jeanne Albert, Alex Friedlander, Barbara Fresko 1oo

3. Codidactic system in the course of mathematics: how to introduce it?
Daniel Alibert 1o9

4. The construction of arithmetic structures by a group of three children across three tasks
Alice Alston, Carolyn A.Maher 117

5. Career choice, gender and attribution patterns of success and failure in mathematics
Miriam Amit 125

6. A classification of students' errors in secondary level algebra
Gerhard Becker 131

7. Teachers'written explanations to pupils about algebra
Joan Bliss, H. Sakonidis 139

8. Algebra - Choices in curriculum design: an exploratory teaching experiment
Alan Bell 147

9. Number naming grammars and the concept of "10"
Garry Bell 154

10. The kindergartners understanding of discrete quantity
Jacques C.Bergeron, Nicolas Herscovics 162

11. A review of research on visualisation in mathematics education
Alan J. Bishop 170

12. Acquisition of meanings and evolution of strategies in problem solving from the age of 7 to the age of 11 in a curricular environment
Paolo Boero 177

13. The relationship between capacity to process information and levels of mathematical learning
Gillian M.Boulton-Lewis 185

14. Mathematical vulnerability
Linda Brandau 193

15. Cognitive psychology and mechanistic versus realistic arithmetic education
Jan van den Brink 201

16. Proof and measurement: an unexpected misconception
Daniel Chazan 207

17. "Discrete" fraction concepts and cognitive structure
M.A./Ken/ Clements, G.A. Lean 215

18. Algorithmic thinking of deaf pupils - report on a three years curriculum project
Elmar Cohors-Fresenborg 223

19. The effect of order-coding and shading of graphical instructions on the speed of construction of a three-dimensional object
Martin Cooper 231

20. New contexts for learning in mathematics
Kathryn Crawford 239

21. Quelques développements récents des recherches sur la discussion autour de problèmes
Jean-Philippe Drouhard, Hélène Lymberopoulou-Fioravantes, Hélène Nikolakarou, Yves Paquelier 247

22. On helping students construct the concept of quantification
Ed Dubinsky 255

23. Children's learning in a transformation geometry microworld
Laurie D. Edwards 263

24. Some cognitive preference styles in studying mathematics
Hamdy El-Faramawy 271

25. Exploring children's perception of mathematics through letters and problems written by children
Nerida F.Ellerton 280

26. The attitudes and practices of student teachers of primary school mathematics
Paul Ernest 288

27. Contexts and performance in numerical activity among adults
Jeffrey T.Evans 296

28. Pre-service teachers' conceptions of the relationships between functions and equations
Ruhama Even 304

29. An experimental study of solving problems in addition and substraction by first-graders
Cao Feiyu, Cai Shanghe 312

30. Beyond ratio formula
Fou-Lai Lin 320

31. Eye fixations during the reading and solution of word problems containing extraneous information: relation to spatial visualization ability
Carol J.Fry 326

32. The meaning of "X" in linear equation and inequality: preliminary survey using cognitive conflict problems
Toshiakira Fujii 334

VOLUME II

III. RESEARCH PAPERS /continued/

33. Intervention in a mathematics course at the college level
Linda Gattuso, Raynald Lacasse 342

34. The education of talented children
Ferenc Genzwein 349

35. The development of a model for competence in mathematical problem solving based on systems of cognitive representation
Gerald A.Goldin 358

36. Pilot work on secondary level
Mária Halmos 366

37. On the textual and the semantic structure of mapping rule and multiplicative compare problems
Guershon Harel, Thomas Post. Merlyn Behr 372

38. Forging the link between practical and formal mathematics
Kathleen M.Hart, A.Sinkinson 380

39. The kindergartners' understanding of the notion of rank
Nicolas Herscovics, Jacques D.Bergeron 385

40. Initial research into the understanding of percentages
Rina Hershkowitz, Tirza Halevi 393

41. Structuring and destructuring a solution: an example of problem solving work with the computer
J.Hillel, J.-L. Gurtner, C.Kieran 402

42. Metacognition: The role of the "Inner teacher"
Ichiei Hirabayashi,
Keiichi Shigematsu 410

43. Formalising intuitive descriptions in a parallelogram LOGO microworld
Celia Hoyles, Richard Noss 417

44. One mathematics teacher
Barbara Jaworski 425

45. Learning the structure of algebraic expressions and equations
Carolyn Kieran 433

46. The influence of teaching on Children's strategies for solving proportional and inversely proportional word problems
Wilfried Kurth 441

47. Constructing bridges from intrinsic to cartesian geometry
Chronis Kynigos 449

48. Concepts in secondary mathematics in Botswana
Hilda Lea 457

49. A developmental model of a first level competency in procedural thinking in LOGO: "May be we're not expert, but we're competent"
Tamara Lemerise 463

50. The naive concept of sets in elementary teachers
Liora Linchevski, Shlomo Vinner 471

51. Concrete introduction to programming languages and observation of Piagetian Stages
F. Lowenthal 479

52. Cognitive and metacognitive shifts
John H.Mason, P.Joy Davis 487

53. Learning mathematics cooperatively with CAI
Zemira Mevarech 495

54. Mathematical pattern-finding in elementary school, focus on pupils' strategies and difficulties in problem-solving
Nobuhiko Nohda 503

55. The construction of an algebraic concept through conflict
Alwyn Olivier 511

56. Gender and mathematics: the prediction of choise and achievement
Wilma Otten, Hans Kuyper 519

57. Teaching and learning methods for problem solving: some theoretical issues and psychological hypotheses
Janine Rogalski, Aline Robert 528

58. Student-sensitive teaching at the tertiary level: a case-study
Pat Rogers 536

59. Strategy choice in solving additions: memory of understanding of numerical relations
Analucia Dias Schliemann 544

60. Representation of functions and analogies
Baruch Schwarz, Maxim Bruckheimer 552

61. Operational vs. structural method of teaching mathematics-case study
Anna Sfard 560

62. Epistemological remarks on functions
Anna Sierpinska 568

63. Formative evaluation of a constructivist mathematics teacher inservice program
Martin A.Simon 576

64. Construction and reconstruction: the reflective practice in mathematics education
Beth Southwell 584

65. Graphical lesson patterns and the process of knowledge development in the mathematics classroom
Heinz Steinbring, Rainer Bromme 593

66. Longer term conceptual benefits from using a computer in algebra teaching
Michael Thomas, David Tall 601

67. The role of audiovisuals in mathematics teaching
Klára Tompa 609

68. Specifying the multiplier effect on children's solutions of simple multiplication word problems
L.Verschaffel, E.De Corte, V.Van Coillie 617

69. Is there any relation between division and multiplication?
Elementary teachers'ideas about division
Shlomo Vinner, Liora Linchevski 625

70. The influence of socialization and emotional factors on mathematics achievement and participation
Delene Visser 633

71. Metacognition and elementary school mathematics
Miriam A.Wolters 641

72. The development of the counting scheme of a five year old child: from figurative to operational
Bob Wright 649

73. Say it's perfect, then pray it's perfect: the early stages of learning about LOGO angle
Vicki Zack 657

74. Substitutions leading to reasoning
Nurit Zehavi 665

IV. LIST OF AUTHORS

RESEARCH REPORTS

(Continued)

INTERVENTION IN A MATHEMATICS COURSE AT THE COLLEGE LEVEL

Linda Gattuso, Cégep du Vieux-Montréal.
Raynald Lacasse, Université d'Ottawa.

Following an investigation conducted with mathophobics students (Gattuso, Lacasse,1986), we formulated a set of working hypotheses for mathematics teaching. We briefly describe how we experimented this pedagogical approach in a regular class of at the college level. The objective was to reconcile affective and cognitive factors.

Not so long ago, a person who aspired to a respectable career had to learn Latin and Greek (Tobias,1980; Giabicani,1985). However today, everyone has to do some mathematics. In Quebec, at the college level, the majority of the students have at least one compulsory or highly recommended mathematics course in their curriculum; source of problems for many, cause of repeated drop out for others. Moreover, upon entering college, these students bring with them at least twelve years of school mathematics. This experience is sometime positive but also too often negative; as confirmed by the failure and drop out rate added to the fact that the students choose their curricula to avoid mathematics (Blouin,1986). Everywhere in the school environment, mathematics are seen as a source of problems and we observed a certain declaration of powerlessness in regard to this question which as many facets: affective, cognitive and behavioral.

A previous study (Gattuso,Lacasse:1986,1987), convinced us of the importance of the affective aspect in the learning of mathematics and brought us to formulate, for the teaching of mathematics, some working hypothesis aiming at alleviating mathophobia. However that is not enough, mathematics courses have substantial content and students with difficulties have gaps in their knowledge which we must address. Likewise some forms of behavior are also a source of difficulty in the learning and the teaching of mathematics (Blouin:1985,1987).

There is no lack of research on the question but teachers coping with all kinds of practical constraints do not succeed in integrating the conclusions of these studies into their practice. Too often these deal only with one particular aspect of the problem and in daily practice many variables interact. To get closer to school reality, it seems important to have a global view of the question. Conscious of this practical difficulty, we tried to articulate and experiment a class intervention model to improve the teaching of mathematics. In a way, we wanted to link the theory and practice so that teachers can easily adapt this model and integrate it subsequently into their own practice.

The starting point

This model was inspired by earlier results (Gattuso, Lacasse, 1986) where the problem of mathophobia was shown to be part of daily life. Nimier (1976), Tobias (1978) and others show the importance of the affective domain. Then, on the grounds of various experiments especially in the United States and some of our own, we put together a supportive environment to reassure some students with a negative background in mathematics: the Mathophobia workshops.

In this research, we wanted to see if there were any changes in the participating students' attitudes and we wanted to identify, the reasons for any such changes. We hope to find a teaching approach that would minimize situations favorable to the appearance of mathophobia. The results and the analysis permitted us to explore different factors on which the teachers could intervene in a regular course of instruction to state some hypothesis along those lines and to group them around four dimensions:

1. Affective aspects vs ability to communicate
2. Peer relations vs learning of mathematics
3. Teacher vs learning of mathematics
4. Pertinence of mathematics.

In short, in addition to listening to the student, the teacher has to allow each individual the opportunity to express his or her own experience of mathematics. The students must have the possibility to exchange, to explore to express orally the processes they use; in order to generate learning. Through his attitudes and his words, the teacher sets out to destroy the myths surrounding mathematics. He must also find occasions to supervise individual learning. He can also show the work inherent in any mathematical process. Some historical references and links with daily experiences will place mathematics in a more human context. To interest and stimulate the students, situations and concrete materials have to be developed.

This calls for a change of behavior on the part of the teacher and this is not easy: he has to be motivated. Instead of being the transmitter of knowledge, the teacher has to support the learning and the work of the students.

This brought us to foresee a second stage in this research; we felt that these ideas had to be tested in a regular class.

In the same vein, Blouin (1985,1987) first developed a group treatment for mathophobia at the college level in Quebec, then studied two more easily detectable phenomena: anxiety and study strategies. Results showed that those who succeed the most are the ones that adopt a more appropriate study behavior and there is also a significant relation between inadequate study behaviors and dysfunctional cognitive reactions, particularly unrealistic beliefs that facilitate the apparition of anxiety and resignation.

Personal factors (other than intellectual aptitude) playing a determinant role in success in mathematics were grouped according to four dimensions:

I. Realistic perception of the necessary conditions to succeed in mathematics
II. Knowing and using adequate working methods
III. See oneself as able to do what is needed to succeed
IV.A sufficient level of motivation (or importance attributed to professional success)

After identifying these differents points, Blouin suggests paths of intervention to undo some erroneous beliefs and permit the development of adequate working behaviors by means of teacher interventions focussing on these points.

In the United States (Sadler et Whimbey:1985), a new experimental approach seeks to improve thinking ability through a holistic approach giving a large place to communication in improving global intellectual operations. It emphasize the fact that learning is an active process and that learners have to participate in the knowledge acquiring process. Six principles support this approach.

1.Teach students to learn in an active way.
2. The students must articulate his thought.
3. Promote intuitive comprehension.
4.Organize the course in a sequential way.
5. Motivate the learners.
6.Establish a social climate favorable to learning.

Remarkable progress in intellectual development, in addition to an increase in motivation and in knowledge, especially in mathematics, were observed.

In France, Claudine Blanchard-Laville (1981), was also interested in students' coping with a handicap in mathematics, in the context of university level course in statistics.. Time allocated for the course was doubled to allow the pace of the course to be that of the students. She used small group work and discussions. At the end of each session, there are some discussion allowing for the verbalization of some affective aspects of learning. This approach demands an important personal investment in terms of work and participation from the student. The content is also modified in a way to provoke active thinking and critics, the objective being to help the student overcome his anxiety while learning to use statistics in an autonomous and constructive way.

Although they issue from different theoretical framework, these experiments converge in many points. A lot of importance is given to communication and particularly to the affective domain. Group work is promoted.

With this in mind we planned three steps to reach the stated objectives.. The first one was to be exploratory to permit a more concrete elaboration of our intervention model. The second one, the experimentation, observation and evaluation of the model in a regular class. Thirdly, we wanted, following the analysis of the intervention, to draw up a realistic pedagogical model that could be use in a regular teaching setting.

Realization

We observed the progress of 2 college groups of 36 students each. The contents of the course was in line with the regular program but followed the hypotheses of the preceding research. These students followed a remediation course which is offered to those who do not have the prerequisits for collegial level courses. The non-homogeneity of the ages and acquired knowledge of these groups complicated the situation. Most of the students were between 17 and 20 years old.

Very quickly, two major differences in regard to the workshops appeared. First, in addition to mathophobia, we found a great indifference towards mathematics, and toward learning in general. Secondly, in a regular class, the gaps in knowledge had a major effect on the subsequent performance of the students. There had to be remediation at the same time for the absence of knowledge, the working behavior and , in general, for skills needed at the process level.

Deficient study behaviors: included low level of persistence, and lack of working autonomy. Students did not feel responsible for their lack of knowledge so they did not assume responsibility for their own work. We found an extraordinary degree of passivity. They are experienced students and unfortunately they have been in contact with aspects of mathematics that have no meaning for them. Work has to be done on two levels: we have to make allowances for the affective and behavioral components, but must also promote progress in knowledge. There was a need to develop stimulating activities, rich in content and to permit the experience of success in mathematics; but first, to overcome a very solidly anchored apathy. A reconciliation between the affective and the cognitive domains had to be provoked. These two objectives were pursued the next session with a group of 25 students.

The course was organized in four blocks.The themes allowed for an exploration of concepts in a concrete or manipulatory way, followed by activities aimed at the development of technical mastery and self confidence. Using general themes offered more potential for giving meaning to mathematical activity.

The initial stage is important, it has to be special. So, for the beginning, the activities aimed to sensitize the students to mathematics work while coping with the affective aspects linked to this work. We used exploration and problem solving situations presented as games, puzzles and geometrical constructions. The first meeting was used to get in touch. After answering an autobiographic questionnaire and an attitude questionnaire, each person introduced him/herself to the others and was able to express personal feelings on mathematics, on fears, on expectations. Moreover, the teacher tied to learn each students' name.

For the other activities, we had to develop a method consistent with our hypotheses, for example: feed-back (discussion in the class on content or the working method), so the student is able to discuss his progress in addition to verbalizing himself on the impressions felt while working; group work, to develop autonomy and taking charge of learning in a supportive environment.

With respect to basic algebra, it was decided not to dedicate time specially for this activity because students do not believe in it. Their sense of helplessness is very clear when you try to submit them to exercises that have already failed. The basic techniques were integrated with others activities so as to give meaning to these formal manipulations.

The study of conics, for example, gives support to all kinds of manipulations and the teacher can draw on the fact that students' interest is sustained by the inherent interest of the forms and their possible applications. Consequently, analytic geometry was our second block. Functions and trigonometry would follow.

Bearing in mind the importance of concrete material in learning, we tried to find supports for the activities. We had to explore, invent and improvise. For the first block, the material used for the workshop was readily available and familiar. For the rest, in addition to usual instruments (protractor, graphic paper, etc...) cardboard and acetates were used for exploratory work centered on manipulation.

The procedure was as follows: following written protocols the students worked in groups at a discovery or problem-solving activity. Explanation on the board followed and was used to bring together the results. In order to be retained, learning of a skill has to be reinforced. The arrangement of the class gradually changed and the pace was that of the students. The teacher continually moved around in the class to observe the students' work, to give support or to refocus activities where needed. This way, the student was able to dominate the situation and to assume responsibility for his own progress. Every task was presented so the student could give meaning to the concepts he constructed. These situations provided the occasion for the student to experience real success in mathematics and for this, it is necessary not to oversimplify the problems ("I have it, but it was easy...")

The autobiographic questionnaire and the attitude questionnaire (completed at the beginning and at the end of the course) gave us some information, but the main information was derived from the teacher's log book and in the students' interview. The analysis is in progress. But it is already possible to say that the experiment is encouraging and prompts us to go on. On the students' part, we observed some remarkable progress They found out what a mathematical activity could be, they succeeded in giving a meaning to what they were doing. The importance of the answer declined, working on the process was emphasized. Even if the questionnaire showed that the students still did not link mathematics to daily occupations, they stopped asking what they were for because they found a certain interest and sometimes even pleasure in this activity. In spite of some stress inevitably connected with any innovation, for example, negative reactions from the students, pressure from the curriculum, the teacher was able to implement this approach which proved highly rewarding. Some students came up with some new problems by themselves, others redid homework already marked, new questions were asked. In this context the class atmosphere was relaxed; at first glance, there could appear to be confusion but the activity was intense and students frequently continued on with their work beyond the end of the end of the period.

This experiment allowed us to implement this approach aimed at reconciling cognitive and affective factors in order to create an enriching mathematical environment. The data analysis will permit us to see which of our objectives were really reached, as well as to reveal the problems of transferability. Next, the replication of this experiment will allow us to produce an improved model adaptable to the regular classroom

REFERENCES

BLANCHARD-LAVILLE, C. (1981). "Les dimensions affectives de l'apprentissage des statistiques", Education permanente, no 61, 41-62.

BLOUIN, Y. (1985). La réussite en mathématiques au collégial: le talent n'explique pas tout. Cégep François-Xavier-Garneau.

BLOUIN, Y. (1986). Réadapter les handicapés des mathématiques. Prospectives, 22 octobre, 115-121.

BLOUIN, Yves (1987). Éduquer à la réussite en mathématiques. Cégep François-Xavier-Garneau.

GATTUSO, L. et LACASSE, R. (1987) Les mathophobes une expérience de réinsertion au niveau collégial. Actes du onzième congrès International de Psychology of Mathematics Education, PME-XI, édité par Jacques C. Bergeron, Nicolas Herscovics, Carolyn Kieran. Juillet Montréal.

GATTUSO, L. et LACASSE, R. (septembre 1986). Les mathophobes une expérience de réinsertion au niveau collégial. Cégep du Vieux Montréal.

GIABICANI, préface de WEYL-KAILEY, L. (1985) Victoire sur les maths, collection «Réponses», Paris: Robert Laffont.

NIMIER, J. (1976). Mathématiques et affectivité. Stock.

SADLER, W., WHIMBEY, A., (1985). A holistic approach to improving thinking skills, Phi Delta Kappan, nov. (Vie Pédagogique, 42).

TOBIAS, S. (1980). Le mythe des maths, traduit par Romain Jacoud. Paris-Montréal: Etudes vivantes.

THE EDUCATION OF TALENTED CHILDREN

Ferenc Genzwein
National Centre for Educational
Technology/Hungary

In recent years we have heard more and more about the education of talented children. It is usually said that the talent must be given an opportunity to develop in school, and the tendency to equalise all children must be changed.

Although I have not been working in a school for several years, I would like to share with you some of my experiences from the time I was a teacher, about the education of talented children, and mark out some ideas and difficulties.

At one time educational statistics only listed the number of pupils who failed at the end of the year, the good ones did not deserve this, let alone the very best. Schools were reprimanded if failures occured in a large number, schools and teachers who tried to foster talents were not adequately appreciated. The schools which were good at taking care of their talented pupils were followed with at least much doubt and suspicion, lack of understanding as recognition. Here I am thinking about schools which produced above average study results for a number of years, whose students won at school competitions in large numbers, and whose students were accepted to higher education very often. These achievements were not highly appreciated but the school was given the nickname of a "racing stable". Morover they were called "distributors of knowledge", "teachers' school", "teaching material centered", "school with an aristocratic concept of quality". As if the distribution of knowledge, teaching the material was not the task of the school, but some source of trouble. This kind of thinking and the resulting action is not lacking something, but it considers the undoubtedly important things unimportant, i.e. the development of talent, skills, knowledge, systematic learning in general and its special methods, although it is evident that without these the school cannot reach its targets.

Special Classes

In reaching these aims special methods and specialised schools and classes play an important role among others, these are very varied all over the world and in Hungary, too.

In Hungary such specialised units were the so called "small" special mathematics classes in secondary grammar schools which were later reorganised as mathematics specialisations. The number of these classes was fairly large, so a large number of pupils had the opportunity of learning mathematics in more hours per week than the average. Special mathematics classes have been organised since 1962, for pupils with a special interest in mathematics. They have nearly 10 classes of mathematics per week. There are such classes in five grammar schools, one each year, and pupils are accepted to them after a successful entrance examination.

Special units within a school, like a special class can be a successful method in the education of children with a gift for mathematics, as facts have proved it, and this is not the same as the well known "school of the excellent", much rather this is one of the criteria of an "excellent school". These two are not only different in their name but they are very different in their principle. In the excellent school excellent teachers work, and educate excellent pupils in different organisational solutions.

Some have an aversion to special classes in schools. They might suppose that only extremely talented children can attend these classes, who might become "one sided" mathematicians. But on the contrary, pupils in these classes like to learn other subjects, too, what' s more, they do it on a higher level than the average. It is an honour and a pleasure for the teacher if he can teach any subject in one of these classes. A large number of these pupils later became very good doctors, economists, engineers, more rarely specialists in the liberal arts. They could become good professionals, among others because in the secondary school they had the opportunity to have an in-depth knowledge in at least one discipline, which is one of the important criteria of developing talent. On the other hand it was also advantageous that neither the teacher nor the pupils looked upon any other subject as unimportant beside mathematics. In these classes pupils not only developed their mental capacities. They managed to reach a harmony between the mind and the soul. They were less likely to merely fulfil instructions like those whose mental capacities were poorer. They could qualify as excellent not only because of their quick mind, but also because of their behaviour, feeling of responsibility and their work in general. They never stopped short before the goal, i.e. they were characterised by higher morale, more responsibility, general culture, and the ability to penetrate deeper into the intricacies of a profession. This experience proves that the education of talented children in special school groups does not supress the formation of a manysided personality. This is proved on the one hand by the career of pupils from these classes, and on the other, by the experience, that good teaching does not only develop the mind, but the feelings and the will as well. All goog teaching is education at the same time, and learning means education of oneself, too. It is proved by many examples that talent and will, talent and strong character put up with each other fairly well in a person, even if they are not always present at the same time and to the same extent. Talented people are not lacking in strong will in the majority of cases, on the contrary they wish to be more active and useful. Talented people can face conflicts and their capability of resistance is better than the average.

All this might suggest, that once we have a large enough number of specialised classes, all our difficulties in developing talent will be over. We only want to say that specialised classes, not only the above mentioned ones, can be one means in the realisation of the aims of the school, in the field of educating talented children as well.

Competitions in mathematics

One of the fields of the realisation of talent might be competitions in mathematics on different levels.

I do not think the opinion which can be read in Köznevelés /1981. 12./, a Hungarian paper on educational policy, leads us into the desired direction: "The atmosphere of competition may have undesirable side effects, different forms of co-operation may weaken, some pupils may be left out from among those who are rewarded or reinforced." Such an opinion urges us in an indirect way to accept the opinion that as competition may be harmful, we should not have any. It disregards the fact that all processes in pedagogy might be harmful. The advice "Let's not do it because it might be harmful" ties our hands. The competitive spirit must be strengthened at school the opportunity to participation must be given to the best, the middling and the weak ones as well. Competition may also be a means of developing one's talent, it may help the pupils use their abilities to the optimum. Good competitive spirit and practice in competition may be a driving force. Care must be taken not to do this wrongly, either. It must be taken into consideration that fear of the competition, prohibiting competition may cause difficulties right opposite to the ones mentioned above. There is one type of fear, that the weaker ones will not receive any recognition, but there is another one, that the very best will not get the recognition which they would deserve in the competitions. Good competition must be a part of the everyday life of the school.

The Eötvös competitions have been organised in Hungary since 1894, every autumn those who were to pass the final exam in secondary school were given the opportunity to show their knowledge in a competition. The best two papers were awarded 100 and 50 gold crowns respectively, and they were published in the paper of the Society. Many mathematicians to become famous later had their first scholarly success here. These competitions were trials of talent as well. Not all talented pupils took part in the competition, but it was proved that those who won were talented. The competition was trustworthly because it built on a relatively small amount of mathematical knowledge, it tested rather the way of thinking, the richness of ideas, the adaptability of the competitor. It is important to know that those who entered, kept preparing for years for this prestigious competition. There is no lower age limit set for the competitors. It has happened several times, that a young person won. In the preparation valuable help was given by "Középiskolai Matematikai Lapok" /Mathematics Paper for Secondary Schools/, started by Dániel Arany in 1894, too, and the Competition Problems in Mathematics, which contained the problems and elegant solutions of the first 32 competitions as well as valuable notes. From 1949 on this went on under the name of "Kürschák József pupils' competition in mathematics".

The book "Competition Problems in Mathematics has been reissued several times since, it is a valuable reading for both the interested pupil and the teacher. The "Arany Dániel pupils' competition" and the "Secondary School Competition" used all the earlier favourable experiences. Both attracted large numbers of pupils, already in 1962 more than three thousand entered each. Nowadays the number is even higher. The International Student Olympics in Mathematics has been organised for many years, too, Hungarian participants have had very good results. It can evidently be put down to the good tratidions in this country, the preparation is also well planned and high standard, and the participants can be selected from a wide circle. The highest level competition is the "Schweitzer Miklós Memorial Competition" first of all for university students, but younger people can also take part as well, sometimes with success.

The above mentioned Középiskolai Matematikai Lapok widely attracts several thousand pupils and several hundred teachers, who all read it regularly. There are problems set in it for several age groups, the pupils send in the solutions and in the next issue the editors publish the solution and the points earned by the pupils. The system of giving points provides a very good method for learning and developing pupils' skills, and the crticles contributed by members of the Hungarian Academy of Sciences are valuable, too.

Tibor Szele established a very good way of education within a school in 1950 in Debrecen. He called these "afternoons of mathematics", and these were higher level than mathematical circles.

Some aspects of talent development in educational policy and pedagogy

When talking about the education of talented children we have to face both aspects of the issue: a complicated problem in social policy and also one of educational policy.

It is a practical problem that talent and genious are sometimes used as synonyms, although beside the number of talented people, i.e. those with an average talent the number of real geniouses is negligible. It is a fact that there have always been geniouses. Gauss already solved difficult mathematical problems at the age of three. Ampère could also calculate at the age of four. Canova was a confectioner's apprentice when his talent for sculpture was already evident: he shaped such an excellent lion of butter that he attracted the attention of a senator in Venice and earned his patronage. Mihály Munkácsy also showed his talent as a painter when painting boxes and the joiner's apprentice became a world-famous painter. László Lovász already wrote good scholarly papers in mathematics when he was in secondary school, he was a student when he got a scholarly degree, and he was just about thirty when he was elected a corresponding member of the Hungarian Academy of Sciences.

Lipót Fejér was 31 years old in 1911 when he became a full professor at the university in Budapest. He was 30 when he formulated a basic thesis in the theory of Fourier lines and thus opened the way to modern analysis. Rossini the famous composer was a lazy boy, so his father apprenticed him to a blacksmith. Davy, one of the pioneers in electronics did not want to learn either.

Schools must draw no consequences from the above things. Least not that they can or should try to educate Gausses or Darwins. They should not think either that the way to the development of talent leads through failure at school or onesided education. But they should not think either that if the school misses out on something the talented pupil will make up for it later anyway.

It is not the geniouses who give us our most trying tasks but the so called typical talents. Geniouses are rare exceptions among people, who might be lucky or unlucky, sometimes a blessing, sometimes a disaster for the society. He always remains an exception, someone extraordinary. The creative capacity of those with an average talent is better than the average or it can be developed to be such. They are able to organise their thoughts and actions better than the average and they are able to cover one or more fields of universal life. Talent understood in this way can be found in the majority of children, and the circumstances /school education first of all in our case/ might unfold it, leave it latent, or make it waste away depending on whether the influences are favourable or not, as teenagers are still changing. So if we apply the adjective "talented" to teenagers it does not mean a state but rather better possibilities for development than the average. That is why the education authorities keep trying to find theoretical and practical solutions of how to educate talented children in an institutionalised form, because the task of education is to promote and urge this development.

Sorry to say schools have not taken into account that different children have different inclinations and abilities, they set the same tasks to everybody. Already at the turn of the century the practice was that a well defined quantity of teaching material had to be taught in previously decided steps. This has basically remained the same up to the present time. The school does its job in a prescribed "order", and the personality of the children can manifest itself only within this framework. With some exaggeration we could say that the centrally defined teaching material is not prepared in view of the child to be taught, but of an age group or of a year in a certain type of school. The stress is laid on the teaching material itself, the textbooks and other teaching aids. There are some new measures though: the teaching material is broken down to basic and additional units, this and specialisation opens the way to changes, but petrified practices hinder the quick changes. So far we have not been able to find the infallible means and methods of how to find and develop talented children, probably they do not exist. But since school practice cannot do without selection and the application of different practices, it must operate so as not to lead the pupil into a dead end, eventually causing tragedy in a period of his life.

It has become evident also that a single rigid central "set of orders" does not work. The most important task seems to be to operate the schools in the framework of an intensification programme as regards the contents and quantity of the teaching material as well as the teaching methods. I do not think about setting up a new type of school when introducing and spreading the theoretical and practical aspects of this concept. This is a collection of modern pedagogical methods and teaching materials, which have been part of earlier experiments. Drawing the consequences from earlier experiences and developing the methods further we can expect higher activity and productivity in schools, that the pupils will do more independent work, their creative ability will grow. Beside presenting knowledge and usual explanations more room will be freed for individual observation, experimentation so that the pupil can be more active in acquiring knowledge. It is not the potential intensity of abilities which is important but the frequency and method of their utilisation. Talent develops through activity.

Teachers must accept the principle that disciplined school life is not a mere conformity with rigid regulations, spirit is not the same as pedantry without ideas, good methods must become common practice.

Those who know our schools from the inside, are aware that although there are a number of tasks to be done, they do their best for establishing themselves as creative workshops. /This seems to be proved by the fact also, that the international society measuring achievements of teachers /IEA/ when measuring such achievements in mathematics and the natural sciences in 24 countries of the world, came to Hungary as well, it found that in the age groups of ten, fourteen and eighteen year-olds the Hungarian pupils were outstanding. In many comparisons they were ahead of their age group. If they were not the first, they were among the best./ We should never forget that the person able to create something great always worked very hard in all walks of life and found the aim and meaning of his life in this work. Gorky put it like this: *talent is work.* We should never get stuck in the bleak practices of the usual, but we must renew ourselves lead by stimulating dissatisfaction ad the wish for becoming more and we must surmount pleasant repetition with constantly seeking for what is new and better.

THE DEVELOPMENT OF A MODEL FOR COMPETENCE IN MATHEMATICAL PROBLEM SOLVING BASED ON SYSTEMS OF COGNITIVE REPRESENTATION

Gerald A. Goldin
Center for Mathematics, Science, and Computer Education
Rutgers - The State University of New Jersey

An overview is provided and theoretical progress reported on the development of a unified model for competence in mathematical problem solving. The model is based on five kinds of mature internal cognitive representational systems: (a) verbal/syntactic; (b) imagistic; (c) formal notational; (d) heuristic planning and executive control; and (e) affective. Three stages of construction are posited: (1) semiotic; (2) structural developmental; and (3) autonomous. New features described in the paper include developmental precursors of imagistic representational systems, and interactions of affective states with heuristic planning configurations. In the present model, the mutual- and self-reference of systems of representation provide an alternate way to understand what have been called metacognitive processes. Implications are drawn for the psychology of mathematics education.

COGNITIVE REPRESENTATIONAL SYSTEMS

In earlier work the author has explored the definition of a representational system (RS), and proposed a model for problem solving competence based on systems of cognitive representation. Briefly, an RS consists of primitive characters or signs somehow embodied, together with rules for forming permitted configurations of these, and for moving between configurations. It also includes higher level structures of various kinds. Configurations or structures in one RS can stand for or *symbolize* those in another. An RS can admit *ambiguities* which are resolved by going outside the system through *symbolization* (Goldin, 1987; Kaput, 1983,1985; Palmer,1977).

Cognitive RS's are constructs. They provide a framework for simulating the internal processing which takes place when people solve mathematics problems, playing the role of "higher level languages" in relation to the possibility of "lower level" descriptions by neural networks. They are intended to describe *competence*--the capability of performance--rather than behavior directly.

Fig. 1 illustrates a narrow, naive model entailing only two RS's--a model too often adopted in teaching mathematical problem solving. The goal in this model is for the student to *translate* the problem directly from its presented form in words and sentences into the formal mathematical notation of numerals, formulas, and equations; and then to manipulate the symbols algorithmically. But the educational objectives--competencies--implicit in a model such as this, are highly inadequate. The translation process can be achieved in many situations by teaching rote "key word" recognition ("altogether" means + , "of" means x , etc.), as if a dictionary procedure were indicated. We regard this as *non-insightful* problem solving. The present model (see Fig. 2) incorporates a much more complex view of what is involved. It is based on five kinds of mature internal cognitive RS's: verbal/syntactic, imagistic systems, formal notational, heuristic planning and executive control, and affective.

A *verbal/syntactic* RS refers to capabilities for processing natural language on the level of words and sentences--through dictionary information, word-word association, parsing of sentences based on grammar and syntax information, etc. *Imagistic systems* refer to non-verbal, internal configurations representing objects, attributes, relations, and transformations. They describe what might loosely be called "semantic" information, and are needed for the meaningful interpretation of verbal problem statements. Here dwell such interesting theoretical constructs as "phenomenological primitives" (diSessa, 1983). The *formal notational systems* of mathematics are highly structured symbolic RS's--numeration systems, arithmetic algorithms, algebraic notation, rules for symbol manipulation, etc. Rather unfortunately, the vast majority of school mathematics today is exclusively devoted to learning their use. In problem contexts, their structure can be explored through state-space analysis (Goldin, 1980). The *heuristic process* (HP) is taken as the culminating construct in an RS of *heuristic planning and executive control*. Four dimensions of analysis have been proposed for examining and comparing HP's: advance planning reasons for using them, domain-specific ways of applying them, domains to which they *can* be applied, and prescriptive criteria for suggesting *that* they be applied (Goldin and Germain, 1983). Finally, an *affective system* describes the changing states of feeling that the problem solver experiences--and utilizes--during problem solving.

STAGES OF CONSTRUCTION

Constructivist researchers argue that knowledge is "constructed" by each individual, rather than "transmitted" or "communicated" (Cobb and Steffe, 1983; von Glasersfeld, 1987). This metaphor can be given a more detailed interpretation by regarding the cognitive RS's in the present model as constructed during three main stages. An *inventive-semiotic stage* incorporates the development of new signs and/or configurations, and the initial acts of symbolization in which these are taken to stand for aspects of a previously established RS (Piaget, 1969). There follows a *period of structural development* for the new RS, driven primarily by structural features of the previously established system. Last, we enter an *autonomous stage*, during which the new RS "separates" from the old. Alternate symbolic relationships now become possible for the new system, enabling the transfer of competencies to new domains.

DEVELOPMENTAL PRECURSORS OF IMAGISTIC SYSTEMS

The above ideas are illustrated by attempting a unified description of the development in children of imagistic RS's from their precursors. One possible conceptualization of such development, generally consistent with Piagetian cognitive-developmental theory, is illustrated in Figs. 3 and 4. Space permits only a brief discussion here. The "brain system" is to be thought of as representing inborn human capabilities. It provides a kind of template for sensory development, facilitating the construction--through sensory-motor feedback, *via* the above stages--of an RS called the "sensory interpreter." This system enables the individual to process sense-data meaningfully, representing for instance the self-other correspondence that makes imitation possible in the child. It in turn serves as the main template for construction of imagistic RS's. This takes place through the principal input and feedback channels shown, again *via* the stages discussed above. The correspondence with Piaget's broad developmental stages is indicated. This picture describes what might be called the "bottom up" development of the imagistic systems which enter into the model of Fig. 2. Later, during the autonomous stage of imagistic cognitive RS's, their development continues by means of verbal, formal, heuristic, and even affective pathways--as well as through new object constructs by way of the channels in the diagram.

AFFECTIVE PATHWAYS

The affect described in the present model is not limited to global attitudes or personality traits such as degree of independence (cf. McLeod, 1985). The emphasis rather is on *local* affect. The changing states of feeling expressed by solvers during mathematical problem solving serve important functions, for experts as well as novices. They provide useful information, facilitate monitoring, and suggest heuristic strategies. Fig. 5 illustrates two major affective pathways, one favorable and one unfavorable, together with conjectured relationships between the affective states and useful or counterproductive heuristic configurations.

METACOGNITION *VERSUS* MUTUAL- AND SELF-REFERENCE

The term metacognition has been used to refer to problem solvers' knowledge *about* their own knowledge states, *monitoring* of their own cognitive processes, or belief systems about problem solving or about themselves. While considerable importance has been ascribed to it in mathematics education (e.g. Schoenfeld, 1983, 1985a,b), there remain serious difficulties in trying to distinguish consistently between the cognitive and the metacognitive. If we acknowledge "objects" to be cognitive constructs, then everyday cognitions about objects are already metacognitive. Tables and chairs, words and sentences, numbers, mathematical formulas and equations, ideas, feelings, and heuristic plans are all commonly treated (and manipulated) as "objects." A heuristic process such as "trial and error" can be applied to "try" objects, numbers, or heuristic plans, and to evaluate the outcomes of the trials.

The present model rejects the cognitive/metacognitive distinction as such, but conjectures explicitly that the *same* cognitive processes can be applied to various domains, consisting of configurations from various RS's. Cognitive RS's are thus *mutually referential*--as when equations (formal notational configurations) serve as "objects" and are manipulated imagistically. They are also *self-referential*--as when words and sentences refer to words and sentences, or heuristic processes act on domains of heuristic processes. This conceptualization allows us to avoid the infinite regress of levels of executive control, and to describe within the model a major complexity of human problem solving.

IMPLICATIONS FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

The model suggests a psychological basis for establishing objectives in teaching mathematical problem solving, and provides a unifying way to interpret concept and schema development, so that these encompass relationships among the indicated cognitive RS's. It thus carries further a line of thinking explored by Lesh (1981) and Lesh, Landau and Hamilton (1983). *Our goal should be to develop in students all of the internal systems of representation, as well as the processes that enable these systems to reference themselves and each other.* Emphasis on formal notational systems only may lead to rote algorithmic learning; exclusive reliance on verbal/syntactic processing may limit students to vocabulary learning and rote translation methods. We must focus explicitly on the development of imagistic systems (including mathematical visualization, kinesthetic encoding, etc.), the executive system (including heuristic processes in all their aspects), and the affective system (including its productive use in monitoring and in the evocation of heuristics). Structured clinical interview research methods are well-suited for investigations based on the present model, which can provide a theoretical framework from which to proceed in aiming for replicability and comparability among such studies.

BIBLIOGRAPHY

J. C. Bergeron & N. Herscovics, eds. (1983). *Procs. of the Fifth Annual Meeting of PME-NA*, Vol. 2. Montreal: Concordia Univ. Dept. of Mathematics.

P. Cobb & L. P. Steffe (1983), Jour. for Res. in Math. Educ. **14**, 34-94.

A. diSessa (1983), in Gentner, D. & Stevens, A. L., eds., *Mental Models.* Hillsdale, N. J.: Erlbaum.

G. A. Goldin (1980), in Goldin, G. A. & McClintock, C. E., *Task Variables in Mathematical Problem Solving*, Hillsdale, N.J.: Erlbaum (orig. Franklin Institute Press), 103--169; --- (1987), in Janvier, *op. cit.*, 59-65, 125-145; --- and Y. Germain (1983), in Bergeron & Herscovics, *op. cit.*, 121-128.

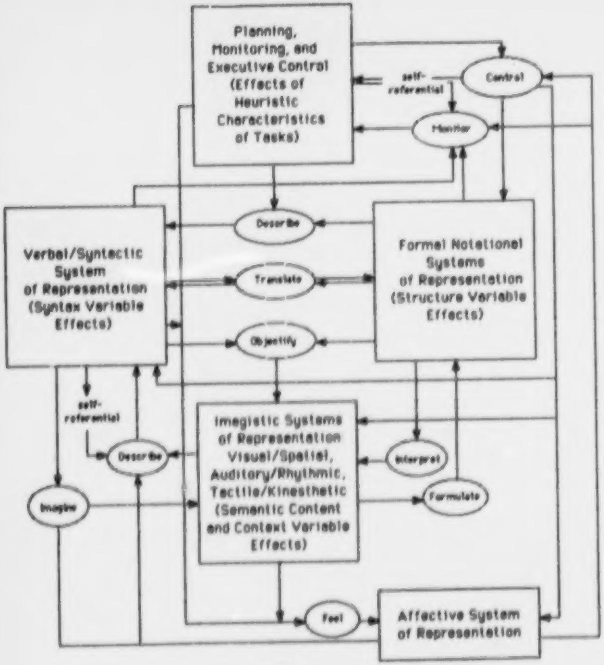

Planning, Monitoring, and Executive Control (Effects of Heuristic Characteristics of Tasks)
self-referential
Control
Monitor
Describe
Translate
Objectify
Verbal/Syntactic System of Representation (Syntax Variable Effects)
Formal Notational Systems of Representation (Structure Variable Effects)
self-referential
Describe
Imagine
Imagistic Systems of Representation Visual/Spatial, Auditory/Rhythmic, Tactile/Kinesthetic (Semantic Content and Context Variable Effects)
Interpret
Formulate
Feel
Affective System of Representation

Figure 2

A Model for Competency in Mathematical and Scientific Problem-Solving

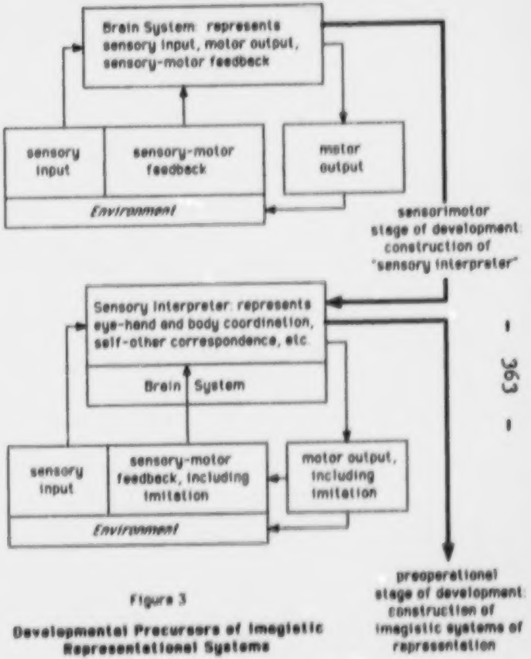

Brain System: represents sensory input, motor output, sensory-motor feedback
sensory input
sensory-motor feedback
motor output
Environment
sensorimotor stage of development: construction of "sensory interpreter"
Sensory interpreter: represents eye-hand and body coordination, self-other correspondence, etc.
Brain System
sensory input
sensory-motor feedback, including imitation
motor output, including imitation
Environment
preoperational stage of development: construction of imagistic systems of representation

Figure 3

Developmental Precursors of Imagistic Representational Systems

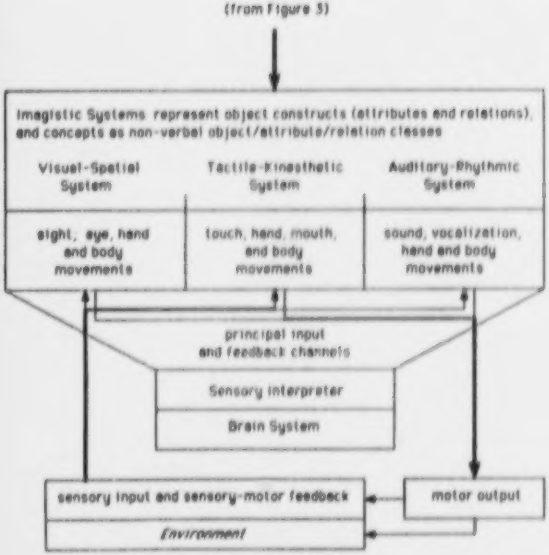

(from Figure 3)
Imagistic Systems represent object constructs (attributes and relations), and concepts as non-verbal object/attribute/relation classes
Visual-Spatial System
Tactile-Kinesthetic System
Auditory-Rhythmic System
sight, eye, hand and body movements
touch, hand, mouth, and body movements
sound, vocalization, hand and body movements
principal input and feedback channels
Sensory interpreter
Brain System
sensory input and sensory-motor feedback
Environment
motor output

Figure 4

Construction of Imagistic Representational Systems

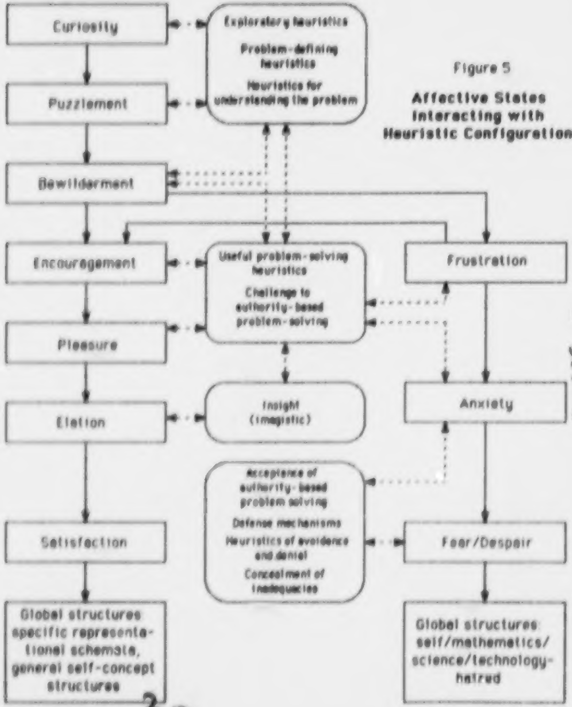

Curiosity
Puzzlement
Bewilderment
Encouragement
Pleasure
Elation
Satisfaction
Global structures: specific representational schemata, general self-concept structures
Exploratory heuristics
Problem-defining heuristics
Heuristics for understanding the problem
Useful problem-solving heuristics
Challenge to authority-based problem-solving
Insight (imagistic)
Acceptance of authority-based problem solving
Defense mechanisms
Heuristics of avoidance and denial
Concealment of inadequacies
Frustration
Anxiety
Fear/Despair
Global structures: self/mathematics/science/technology-hatred

Figure 5

Affective States Interacting with Heuristic Configurations

BEST COPY AVAILABLE

C. Janvier (1987), ed. *Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, N.J.: Erlbaum.

J. Kaput (1983), in Bergeron & Herscovics, *op. cit.*, 57-66. --- (1985), in Silver, *op. cit.*, 381-398.

R. Lesh (1981), Educ. Studies in Math. **12**, 235-264. --- , M. Landau, and E. Hamilton (1983), in Lesh & Landau, *op. cit.*, 263-343.

Lesh, R. & Landau, M., *Acquisition of Mathematics Concepts and Processes*. New York: Academic Press.

D. McLeod (1985), in Silver, *op. cit.*, 267-279.

S. E. Palmer (1977), in Rosch, E. & Lloyd, B. B., eds., *Cognition and Categorization*, Hillsdale, N. J.: Erlbaum.

J. Piaget (1969). *Science of Education and the Psychology of the Child*. New York: Viking Press.

E. A. Silver (1985), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*. Hillsdale, N. J.: Erlbaum.

A. Schoenfeld (1983), in Lesh & Landau, *op. cit.*, 345-395; --- (1985a), in Silver, *op. cit.*, 361-79; --- (1985b), *Mathematical Problem Solving*. New York: Academic Press.

E. von Glasersfeld (1987), in Janvier, *op. cit.*, 3-17.

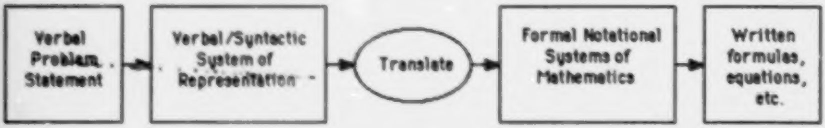

Verbal Problem Statement
Verbal/Syntactic System of Representation
Translate
Formal Notational Systems of Mathematics
Written formulas, equations, etc.

Figure 1

A Common Conception of Mathematical Problem Solving

PILOT WORK ON SECONDARY LEVEL

Mária Halmos
Hungarian National Centre for Educational Technology (OOK)

ABSTRACT

A new mathematical educational project on primary level was initiated in Hungary in 1962 according to the conception of Tamás Varga.A new programme based on this project has been prepared which was implemented during the seventies and early eighties. Pilot work started on secondary level during the early seventies. The objectives of mathematical education in the new primary programme and in the secondary experimental programme are essentially the same. The secondary experiment is projected and guided by the members of the Mathematical-Didactical Research Group of OOK (Hungarian National Centre for Educational Techlogy). The most important educational objectives of the experiment are the following: learning should begin at a very concrete starting point and then lead children towards abstraction; learn mathematics through the discovering of mathematics; make mathematics liked by children.

1. INTRODUCTION

I could not start more adequately than by quoting Tamás Varga about the main motivation of the changing of the Hungarian mathematical education:

> "...Any normal child is able to comprehend and learn every piece of mathematics which is now taught at school, as well as a good deal more, to enable him to fit in with the requirements of modern trends..."

A new mathematical educational project on primary level was initiated in Hungary in 1962 according to the conception of Tamás Varga. A new programme based on this project has been prepared which was implemented during the seventies and early eighties.

Pilot work extending this project to the secondary level has started during the early seventies. The secondary programme was based earlier on the traditional primary programme. A secondary experiment has been started in 1976, which is based on the new primary mathematical programme, modern both in method and in content. The objectives of mathematical education in the new primary programme and in the secondary experimental programme are essentially the same. Differences result from the fact that we face a different age-group.

The secondary experiment is projected and guided by the members of the Mathematical-Didactical Research Group of OOK (Hungarian National Centre for Educational Technology). This research group was directed by Professor János Surányi for more than two decades.

This secondary experiment will be discussed in this report.

The experiment is running in 40 classes with 10-10 classes in the same age-group (age 14-18) at present. The experiments begin in the first classes of secondary school (age 14) continued in each case with the classes of higher grades up to maturity (age 18). Materials are prepared both for children and teachers, and these are changed on the basis of classroom-experiences. These changes are not finalized yet.

2.EDUCATIONAL OBJECTIVES AND PRINCIPLES OF THE EXPERIMENT

a.The abstraction process

The process of abstraction is one for which considerable time must be allowed. Children do not abstract automatically. Mathematics is very abstract. This is in fact its greatest strength, since it means, in effect, that it condenses into itself the essence of a great number of concrete phenomena. In order to get this very abstract structure we have to begin at a very concrete starting point and then lead children towards abstraction.

b.Discovery

"The best way to learn anything is to discover it by yourself", wrote George Polya. This is very true in mathematical learning. This means that there is hardly anything more important, than leading the children to meet mathematics in status nascendi or to make them rediscover it.

When teaching is going this way, learning is realised through problem solving. Definitions, axioms, notations, terminology are also very important in mathematics. Children learn independently to name and symbolize mathematical entities. These ones are also discoverable.

c.Motivation

Learning is based on intrinsic rather than extrinsic motivation. This simply means to try as far as possible to build on children's interest: to provide children with challenging problems which are neither too difficult nor too easy for them; to make children get used to checking and correcting their own work (something is correct not because the teacher has said but because it has been checked and found correct).

Consequently, it is important to consider the individual differences between children. This is, of course, intimately connected with the principle "to discover by yourself". The children that are able to do more will produce more, both in quantity and quality. Another consequence is that nobody should be ashamed of having committed a mathematical error. To have committed an error gives an opportunity for discussion and could never be used by a teacher for making a child look small.

The greatest intrinsic reward for children is to get on happily with the topic in hand. That is connected with the very important objective, to make mathematics liked by children. The mechanical tedious training should be avoided for that purpose.The training is to be embedded into challenging activities.

d.Mathematics as a tool; mathematics as a whole; mathematics as an art.

In learning mathematics application is the best starting point. When it is possible it is worth taking problems from other (nonmathematical) subjects. Problems drawn upon real-life situations may help to develop in children a feeling for order of magnitude and reasonable approximation, skill in estimation or in rapid rough calculation of numerical results.

The structure of the curriculum aims at removing the fragmentation of the various mathematical disciplines: arithmetics, algebra, geometry, and function are integrated in our experimental mathematical documents; all interlacing with each other.

Many children leave school without ever having felt the beauty of mathematics. To make children realise the beauty of mathematics the first step is to remove the fear and anxiety from mathematics. To realise and enjoy the beauty of mathematics children must be given sufficent opportunity for free, creative activity.

3. EXPERIMENTAL DOCUMENTS

The children use the following books:

1. grade (age 14-15, 4 lessons per week)
    - Miscellaneous problems
    - Arithmetics
    - Algebra I.
    - Geometry I.
    - Functions
    - Algebra II.
    - Geometry II.
    - Combinatorics

2. grade (age 15-16, 4 lessons per week)
    - Quadratic function
    - Algebra
    - Trigonometry I.
    - Geometry
    - Trigonometry II.
    - Combinatorics

3. grade (age 16-17, 3 lessons per week)
    - Miscellaneous problems
    - Extension of the concept of power and logarithms
    - Vectors
    - Coordinate-geometry

4. grade (age 17-18, 3 lessons per week)
    - Miscellaneous problems
    - Series
    - Spatial geometry
    - **Recapitulation**

a. Miscellaneous problems

This kind of books is available for first, third and fourth grades. These books contain different kinds of not too difficult, yet non-routine problems.

They usually do not require much more than logical thinking or some unusual combination of simple knowledge. Also mathematical recreational problems can be found amongst them. This kind of problems have a role in helping children to like mathematics. Puzzles can be excellent starting points for deep ideas in school mathematics.

The miscellaneous problems often throw light to some topics of elementary mathematics not treated systematically.

They may also be simple special cases of advanced problems usually discussed in higher mathematics.

Other miscellaneous problems are destined for preparing topics to be treated in details later on.

b. Recapitulation

This book includes concepts, theorems and their proofs, problems and their solutions selecting some topics (sets, arithmetics, algebra, functions, combinatorics) of the 4 years.

c. The structure of textbooks

The textbooks consist of problemseries, which allow the children to discover the subject-matter, then summary of the subject, after that further problems and interesting parts from books and articles concerning the discussed themes.

d. The guides for teachers

The textbooks for children may be discussed according to the order of the listing above, but it is only one possibility. Other possibilities are given in the guides for teachers.

The guides for teachers includes also the solutions of the problems contained in the textbooks and here is listed the problems to be used for the gifted children.

BIBLIOGRAPHY

M.Halmos-T.Varga: Change in Mathematics Education since the Late 1950's -Ideas and Realisation (Educational Studies in Mathematics 1978/9)

G.Polya: Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving, 1962

W.Servais-T.Varga: Teaching School Mathematics (UNESCO Source Book, 1971)

J.Surányi: Remarques sur les tâches de l'enseignement des mathématiques et ses obstacles (Colloque International UNESCO, 1968)

T.Varga: Notes on Mathematics Reform in Hungarian Schools, 1968

ON THE TEXTUAL AND THE SEMANTIC STRUCTURE OF MAPPING RULE AND MULTIPLICATIVE COMPARE PROBLEMS

Guershon Harel
Northern Illinois University
Thomas Post
University of Minnesota
Merlyn Behr
Northern Illinois University

In this paper we will (a) compare the textual and the semantic structures of division problems in the Mapping Rule category to those in the Multiplicative Compare category, (b) show how different interpretations of the string underlying the textual structure of Multiplicative Compare problems -- the phrase "as many as" -- influence the representation of division problems as partitive or quotitive, and (c) suggest an instrument to answer empirically the question of what implicit interpretation students give to the phrase "as many as."

In analyzing the propositional structure of multiplicative problems, Nesher (1987) identified and formulated three different categories: Mapping Rule, Multiplicative Compare, and Cartesian Multiplication. In this paper we are interested in the textual and the semantic structures of the first two categories.

Mapping Rule. In a Mapping Rule problem there is a mapping rule between the two measure spaces from which the units are derived. For example, in the multiplication (M) problem:

M_1. There are 5 shelves of books in Dan's room.
Dan put 8 books on each shelf.
How many books are there in his room?

This research was supported in part by the National Science Foundation under grant No. 44-22968. Any opinions, findings, and conclusions expressed are those of the authors and do not necessarily reflect the views of National Science Foundation.

the phrase "8 books on each shelf" is the mapping rule between the measure spaces "shelves" and "books."

Nesher characterized the two types of division problems in the Mapping Rule category, partitive and quotitive, as follows. A division problem is partitive if the question is about the string which was the mapping rule in the corresponding multiplication problem, such as in the following division (D) problem:

D_2. There were 40 books in the room, and
5 shelves.
How many books are there on each shelf?

A division problem is quotitive if the question is about the string which was an existential description in the corresponding multiplication problem, such as in the following division problem:

D_3. There were 40 books in the room.
8 books on each shelf.
How many shelves were there?

Multiplicative Compare. A Multiplicative Compare problem is one in which a one-directional-scalar-funciton is used to compare between two problem quantities. For example, in the multiplication problem

M_4. Dan has 12 marbles.
Ruth has 6 times as many marbles as Dan has.
How many marbles does Ruth have?"

the phrase "Ruth has 4 times as many marbles as Dan has" is the one-directional-scalar-function between the quantities representing Dan's set of marbles and Ruth's set of marbles.

Nesher did not characterize partitive and quotitive problems in the Multiplicative Compare category. However, according to Greer's (1985) extension of the type of division problems, a problem is partitive or quotitive, respectively, according to whether the divisor is conceived of as the multiplier or as the multiplicand in the corresponding multiplication problem. If we hold that the numbers 6 and 12 in Problem M_4 are the divisor and the multiplicand, respectively, then based on Greer's extension, the following division problems (D_5 and D_6) would be partitive and quotitive, respectively. (As can be seen from Problems M_1, D_2, and D_3, Greer's extension agrees with Nesher's characterization of Mapping Rule division problems.)

D_5. Ruth has 72 marbles.
Ruth has 6 times as many marbles as Dan has.

How many marbles does Dan have?"

D_6. Ruth has 72 marbles.

Dan has 12 marbles.

How many times as many as Dan does Ruth have?

Using Nesher propositional terminology, we get that a division problem from the Multiplicative Compare category is partitive if the question is on the string which was an existential description in the corresponding multiplication problem (see, for example, division problems D_5 with respect to the multiplication problem M_4). similarly, a division problem is quotitive if the question is about the string which was the one- directional-scalar-function in the corresponding multiplication problem (see, for example, Problems D_6 with respect to Problem M_4).

We will see now that these definitions of partitive and quotitive Multiplicative Compare problems are based on a specific interpretation of the phrase "as many as;" a different interpretation of this phrase would lead to opposite definitions. Consider, for example, Problem D_5. The phrase "Ruth has 6 times as many marbles as Dan has" can be interpreted as a unit-rate-per-statement, i.e., for each marble of Dan, there are 6 marbles of Ruth (see Figure 1), or as a lot-per-statement, i.e., for Dan's set of marbles there are 6 sets of marbles of Ruth, each of which is equivalent to Dan's set (see Figure 2).

Figure 1

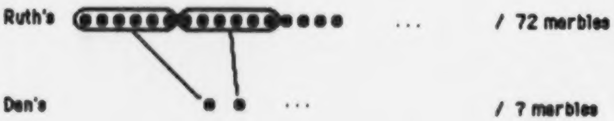
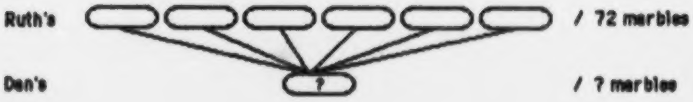


Figure 2



If the phrase "as many as" is interpreted as a unit-rate-per-statement, then Problem D_5 would be conceived of as a quotitive and not as a partitive as was indicated earlier. This is because under this interpretation, to find how many marbles Dan has, one needs to find the number of times the set of 6 marbles goes into the set of 72 marbles (see Figure 1). On the other hand, if the phrase "as many as" is interpreted as a lot-per-statement, the problem situation would suggest that (a) there is one set of marbles belongs to Dan, which against it there are 6 sets of marbles belong to Ruth, each of which is equivalent to Dan's set, and (b) Ruth has 72 marbles. (See Figure 2.) Thus, to find how many marbles Dan has, one needs to find the number of marbles in each Ruth's set. This situation suggests that Problem D_5 is of partitive division type.

Applying the same analysis to Problem D_6, it will be found that the problem is conceived of as partitive or quotitive according to if the phrase "as many as" is interpreted as a lot-per-statement or as a unit-rate-per-statement, respectively.

RELATIONSHIPS BETWEEN PROBLEM STRUCTURES

We indicate that under the lot-per-statement-interpretation, partitive (quotitive) Mapping Rule problems have the same textual structure as the quotitive (partitive) Multiplicative Compare problems (see Figure 3): The question in a Mapping Rule *partitive* problem and in a Multiplicative Compare *quotitive* problem is about the string which was an association (i.e., either as a mapping rule or as a one-directional scalar- function) between two measure spaces in the corresponding multiplication problem. similarly, the question in the Mapping Rule *quotitive* problem and in the Multiplicative Compare *partitive* problem is about the string which was an existential description in the corresponding multiplication problem. On the other hand, under the unit-rate-per-statement interpretation, the Mapping Rule partitive and quotitive problems are of the same structure as of the Multiplicative Compare partitive and quotitive problems, respectively (see Figure 3).

Figure 3

ED - Existential Description **Ass** - Association

Mapping Rule

Multiplication	Partitive	quotitive
- 5 shelves of books (ED)	- 40 books (ED)	- 40 books (ED)
- 8 books on each (Ass)	- 5 shelves (ED)	- 8 books on each (Ass)
- How many books? (ED)	- How many on each (Ass)	- How many shelves (ED)

Multiplicative Compare
(Lot-per-statement interpretation)

Multiplication	quotitive	Partitive
- Dan has 12 marbles (ED)	- Ruth has 72 marbles (ED)	- Ruth has 72 marbles (ED)
- Ruth has 6 times as many marbles as Dan has (Ass)	- Dan has 12 marbles (ED)	- Ruth has 6 times as many marbles as Dan has (Ass)
- How many marbles does Ruth have? (ED)	- How many times as many as Dan does Ruth have? (Ass)	- How many marbles does Dan have? (ED)

Multiplicative Compare
(Unit-rate-per-statement interpretation)

Multiplication	Partitive	quotitive
- Dan has 12 marbles (ED)	- Ruth has 72 marbles (ED)	- Ruth has 72 marbles (ED)
- Ruth has 6 times as many marbles as Dan has (Ass)	- Dan has 12 marbles (ED)	- Ruth has 6 times as many marbles as Dan has (Ass)
- How many marbles does Ruth have? (ED)	- How many times as many as Dan does Ruth have? (Ass)	- How many marbles does Dan have? (ED)

AN EXPERIMENT

We will suggest now an experiment to answer empirically the question of whether the phrase "as many as" in division problems from the Multiplicative Compare category is interpreted implicitly by students as a unit-rate-per-statement or as a lot-per-statement. This experiment is part of an instrument we have developed to assess the inservice teachers' knowledge of multiplicative structures, which is under way and will be reported at Post Harel and Behr (in preparation). Items from this experiment include the following example. We gave students two variations of a division problem. In the first variation the problem quantities violate the intuitive partitive model but conform with the intuitive quotitive model (Fischbein, Deri, Nello, and Marino, 1985). This variation can be achieved, for example, by taking the divisor to be a fractional number and smaller than the dividend. The second variation is a problem in which the quantities conforms with the two intuitive models, which can be achieved, for example, by taking the divisor a whole number and smaller than the dividend. Examples of these variations are Problems D_7 and D_8, respectively.

D_7. Steve has 72 pizzas.
Steve has 6. 3 times as many pizzas as John.
How many pizzas does John have?

D_8. Steve has 72 pizzas.
Steve has 6 times as many pizzas as John.
How many pizzas does John have?

Fischbein et al. (1985) and others (Greer, 1985; Greer and Mangan, 1984; Mangan, 1986; Tirosh, Graeber, and Glover 1986; Harel, Post, and Behr, in preparation) found that children and teachers as well select a non-correct operation when they are presented with problems including numbers that conflict with the rules of the primitive models; students' performance on problems which conforms with the intuitive models is relatively high. Thus, if the phrase "as many as" is interpreted by the students as a lot-per-statement, then, as has been shown earlier, the two variations (D_7 and D_8) would be represented as partitive division problems. Consequently, it would be expected that the students will perform better on the second variation (Problem D_8), which does not violate the partitive model, than on the first variation (Problem D_7), which does violates the partitive model. On the other hand, if the problem is interpreted as a unit-rate-per-statement, then the problem (in the two variations) would be represented as a quotitive division. Consequently, it would be expected that the students' performance would be equally high on the two variations, since both problems do not violate the intuitive quotitive models.

BEST COPY AVAILABLE

CONCLUSIONS

From this analysis we see that the interpretation of the phrase "as many as" affect the semantic structure of Multiplicative Compare division problems. The pedagogical value of this analysis is that it points out the need to enrich the cultural and educational experiences which underlie children's understanding of Multiplicative Compare division problems. Students should be able to move from one interpretation to another in order to construct the problem representation that most incorporates with their knowledge. Our analysis of Missing Value Proportion Problems (Harel and Behr, 1988) and research by many others (e.g., Davis, 1984; Greeno, 1983; Behr, Lesh, and Post, 1986) demonstrate the importance of the use of different problem representations during the course of a problem solution.

The types of the quantities, discrete or continuous, involved in the problem seem to have an impact on the interpretation of the phrase "as many as," and consequently on the semantic interpretation of the problem as quotitive or partitive. As was shown earlier, an "as many as" phrase which involves discrete quantities can be interpreted either as a unit-rate-per-statement or as a lot-per-statement. On the other hand, if the quantities are continuous, it is more likely that the phrase "as many as" would be interpreted as a lot-per-statement, such as in the phrase "a mountain range is 124 times as long as a mural of it." However, this hypothesis and the analysis described in this paper needs further considerations.

REFERENCES

Behr, M., Lesh, R., & Post, T. (1986). Representations and translations among representations in mathematics learning and problem solving. In C. Janier (Ed.). Problems of Representation in the Teaching of Mathematics. Hillsdale, NJ: Erlbaum.

Davis, R. B. (1984). Learning Mathematics: The Cognitive Science approach to Mathematics Education. Norwood, NJ: Ablex Publishing Co.

Fischbein, E. Deri, M., Nello, M. & Marino, M. (1985). The rule of implicit models in solving verbal problems in multiplication and division. Journal of Research in Mathematics Education. 16, 3-17.

Greeno, J. G. (1983). Conceptual entities. In D. Gentner A. Stevens (Eds.). Mental Models (p. 227-252). Hillsdale, NJ: Erlbaum.

Greer, B. (1985). Understanding of arithmetical operations as models of situations. In J. Sloboda and D. Rogers (Eds.) Cognitive Processes in Mathematics. London, Oxford University Press.

Greer, B., & Mangan, C. (1984). Understanding multiplication and division. In T. Carpenter and J. Moser (Eds.). Proceedings of the Wisconsin Meeting of the PME-NA. University of Wisconsin, Madison, 27-32.

Harel, G., & Behr, M. (1988). Structure and hierarchy of missing value proportion problems and their representations. Journal of Mathematical Behavior (in press).

Harel, G., Post, T. & Behr, M. (in preparation). An assessment of inservice middle school teacher's knowledge of multiplication and division problems.

Mangan, C. (1986). Choice of operation in multiplication and division word problems. Unpublished doctoral dissertation, Queen's University, Belfast.

Nesher, P. Multiplicative school word problems: Theoretical approaches and empirical findings. Paper presented at the meeting of the Working Group on Middle School Number Concepts, Northern Illinois University, DeKalb, Il, May, 1987.

Post, T., Harel, G., & Behr, M. (in preparation). Middle school teachers knowledge of middle school mathematics.

FORGING THE LINK BETWEEN PRACTICAL AND FORMAL MATHEMATICS

K. M. Hart and A. Sinkinson
Nuffield Secondary Mathematics

The research project 'Children's Mathematical Frameworks' monitored the transition from the use of concrete materials to the mathematical formalisation which was the synthesis of these practical experiences. The data were collected from interviews with children and classroom observations of teachers and pupils. The children tended to say that there was little or no connection between the two types of mathematical experience. The Nuffield project uses the same methodology as CMF but attempts to compare results and children's opinions when a) the 'normal' transition is made and b) a distinctive, different type of experience is provided to establish a link between the concrete work and formalisation.

The research project 'Children's Mathematical Frameworks' (CMF) was designed to monitor the transition from the use of practical/concrete material to formal/symbolic mathematics. The sample was composed of British children aged between 8 and 13 years. For at least 20 years the training of teachers in Britain has been influenced by the theories of Piaget and more significantly by those who sought to implement these theories in suggestions for classroom practice. Thus teachers have come to believe that the most beneficial method of teaching mathematics to children deemed to be at the concrete operational level of cognitive development, is through the use of concrete materials (manipulatives). This is often extended to 'mathematics should be taught through practical work'. The mathematics taught in the secondary school, however, assumes the use of symbols and generalisations which constitute a more formal system. CMF was only concerned with the use of materials in a series of experiences carefully structured by the teacher so that they culminated in a generalisation, formula or rule. This ' formalisation' was supposed to be the statement or symbolisation of the synthesis of the experiences. For example the formula for the area of a rectangle could be regarded as coming naturally from many activities which involved using tiles, counting squares, drawing shapes to encompass a given number of squares etc.

The methodology of CMF involved the interviewing of some of the children being taught in this way (i) before the start of teaching, (ii) just before the lesson in which the 'formalisation was verbalised, (iii) immediately after this lesson and (iv) three months later. The 'formalisation' lesson(s) were observed and tape-recorded by the researchers. The results, previously reported at PME, showed generally that the pupils saw no connection between the two types of work they had experienced, to quote - 'Sums is sums and bricks is bricks'. Additionally teachers made little effort to describe why the transition was being made nor to emphasise the generalisability of the new 'formalisation'. They might appeal to the time saved by using the new method or to the inconvenience of carrying bricks in order to perform mathematical calculations but generally, although the pupils were told they would understand better if they used materials, the link to a final formula was not stressed.

In September 1987, we were funded by Nuffield for one year to continue and extend this work. The hypothesis for the new research is that the gap between the use of concrete materials and the formalisation (which is often written symbolically) is very great and that children would benefit from a third type of experience, essentially different from both but acting as a bridge between them. This bridge activity might be discussion, child verbalisation, diagrammatic representation, tabulation etc., but its role is clearly seen as that of connecting practical work to more formal mathematics.

The information obtained from the observation of classrooms and teachers during the CMF project, has proved to be very illuminating. The CMF records were of teachers using their own schemes of work and the methods they suggested as effective. These results give a different view of classroom happenings than those reported by mathematics educators who are seeking to change practice by the introduction of innovative procedures. The role of the researcher in the formalisation lessons was that of observer and recorder and was not concerned with intervention. We have sought to extend these data by again observing experienced teachers, audio-taping their words and then transcribing the tapes. The Nuffield research involves nine secondary teachers (seven of whom are heads of mathematics departments) and two primary school teachers, one of whom is the head of the school. Seven of these eleven teachers are engaged in study for a masters degree in mathematics education. Their analysis of the research experiences forms part of the work to be assessed for the award of the degree. The teachers have volunteered to be part of the research and are both experienced and confident. Each is asked to teach one of the following topics (already investigated in CMF) to children for whom they thought it appropriate: (i) the formula for the area of a rectangle, (ii) the formula for the volume of a cuboid, (iii) the rule for generating equivalent fractions, (iv) a method for solving algebraic equations. The rule is to be the synthesis of a series of practical experiences. The teacher chooses two matched sets of children to teach, either the two halves of a class or two classes which are seen as roughly comparable in attainment. One group is taught using concrete experiences leading to a formalisation and the other group has an additional 'bridging' experience.

The teacher gives a pre-test on the topic to each group, the test is provided and marked by the researchers and is based on questions tried and found informative in CMF and CSMS. For the 'normal' group the teacher writes a scheme of work, a copy of which is sent to the researchers for information. The 'formalisation' lesson is tape-recorded and observed by a researcher, then transcribed and analysed. The teacher is provided with a post-test and asked to interview six children in the group in order to amplify the information obtained from it. Some training in interview techniques is given to each teacher and they are supplied with questions to use in the interviews. After the teaching of Group 1 is finished, the teacher meets with the researcher to discuss the nature of the 'bridge' which forms the distinctive feature of the second teaching sequence. The 'bridge' is defined as essentially different to a) the two types of experiences already in the scheme of work and b) whatever was used by the teacher in the 'formalisation' lesson, thus if the teacher used diagrams in the 'formalisation' lesson then diagrams could not be the distinguishing feature of the 'bridge'. The second teaching sequence also includes pre and post tests and interviews as before but has an extra set of activities which form the 'bridge'. Finally the researchers interview children from both sets three months later to discover whether the conscious effort of the teacher to link two very different types of working has helped the children see a link.

The research is in progress (Jan88) and it is hoped that some results can be presented at PME 12. Scrutiny of the schemes of work of five teacher and transcripts of some of their lessons leads us to make some comments which might be recognised as true of other teachers in other places. Firstly, the concrete material is not taken seriously by the teacher in that its essence, be it wood or tin of particular length or weight is often ignored or distorted. For example, one teacher asked a boy how he would show 2x + 3 = 17 using Cuesenaire rods (these were the manipulatives). There is no rod designed to be 17 units in length, so the child is forced to pretend. The conversation continues thus:

[T: Teacher. P: Pupil]

T: (repeats), now how would I do it with my rods? How would I do it with my rods?
P: Put a... say you had blue, on the bottom
T: Put a blue on the bottom, what's that going to represent?
P: The 17
T: That's going to represent the whole lot, the 17.
P: And then say, take pink and that would represent the three that you're taking away
T: That's the 3 I'm taking away
P: And the gap left is the 2x

Already the bricks are superfluous and possibly stand in the way of understanding since their colour and length have no relation to the numbers they are meant to represent. The words 'take away' which convey 'removal' cannot be accurately used if there is no way the requisite amount of wood in the blue rod can be removed. The model set up demonstrates a 'difference' in length. Another teacher also used rods to introduce solution of equations and even when the wood was no longer there, referred to 'chopping'. The child was asked to remember how 2x = 10 was represented:-

T: However we did have some that looked like this, where we had 2 of the rods put together, equalled one whole rod, remember? How did we do with those? Yes, Tamsin

P: Say you had one x there, we put 2x plus.... you split it in half

T: We split it in half, remember, we chopped it in half. We kept takling about chopping it in half, yes? So what did we write down there Christopher, can you remember?

T: 10....

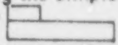
P: Take away 5

T: Oh no, I don't think...I can see you can see it's 5 yes...go back to you Tamsin

P: 10 divided by 2

T: 10 divided by 2, cos we're chopping that one in half, alright? Because we've got 2 of them remember, in your mind, the two rods side by side equals the ten, chop it in half....x equals 10 over 2, x equals....

It is however much more sensible to remove or 'take away' one of the 'x' rods (or 5) than to chop with a non-existing chopper where there is already a split!

Secondly, the material set up to represent the mathematics, very often represents only the simplest case or perhaps only one aspect of the rule. For example if  represents $2x + 3 = 9$, how does one represent $2x - 3 = 9$? Consequently, much of the formalisation is based itself on a formalisation which is tied to the material. This does not deter teachers from referring (verbally) to the manipulatives although the mathematics being discussed cannot itself be represented by them. A classic example is referring to -3, -4 as points on a number line when the topic under consideration is multiplication of negative numbers.

It is possible that by trying to 'make concrete' certain parts of mathematics, we have confused rather than helped children. Can teachers be expected to set up concrete models for many topics, in such a way that they cover a number of situations and not just the simplest? In our research we hope to provide evidence of planned ways of bridging the gap between concrete experiences and formalisation in situations where the teacher thought the practical aspect would be effective.

THE KINDERGARTNERS' UNDERSTANDING OF THE NOTION OF RANK

Nicolas Herscovics, Concordia University
Jacques C. Bergeron, Université de Montréal

Abstract

When the natural numbers are viewed as the means to measure the rank of an object in an ordered set, the notion of rank can then be considered as a pre-concept of number. This paper reports the results of a study regarding the kindergartners' understanding of rank. Our investigation shows that three distinct components of understanding can be found among this age group. All 24 children tested indicated they had an intuitive understanding as evidenced by their ability to estimate order related concepts on the basis of visual perception. A more advanced level of comprehension, that of procedural understanding, was evidenced when each child proved able to use procedures based on one-to-one correspondences to construct ordered sets subject to constraints on some elements which had to be positioned before or after or together with another one. A third component of understanding, that of abstraction, was studied through various tasks ascertaining the subjects' ability to perceive the invariance of rank with respect to various surface transformations, that is, changes in the disposition of the objects which did not affect the given rank

In their seminal study on the emergence of number in the child's mind, Piaget and Szeminska (1941/1967) discriminated between the cardinal and ordinal aspects of number. Much of their work on cardinality was an extension of earlier work involving the conservation of liquid and mass. They approached ordination through the study of asymmetric relations such as those implied in the seriation of objects of different lengths or of different masses. Thirty years later, Brainerd (1973) sought to establish a possible priority between the two complementary aspects of number also by using tasks involving length and mass and found that "ordination emerges prior to cardination". However,his findings may be attributed to his exprimental design for they are in sharp contrast with Piaget's own conclusions:

> "Several authors (Freudenthal,etc.) seem to have understood that I think the ordinal number is more primitive than the cardinal number, or the opposite. I have never made such a statement and have always considered these two aspects of finite numbers indissociable and psychologically reinforcing one another in a synthesis that goes beyond both the inclusion of classes and the order of asymmetrical transitive relations" (Piaget,1973,p.82)

- We wish to thank our research assistants Anne Bergeron and Marielle Signori whose suggestions have improved the quality of both the tasks and the questions.
- Research funded by the Quebec Ministry of Education,FCAR Grant EQ 2923

But in fact, Freudenthal's disagreement with Piaget runs much deeper. It is at the level of fundamental definition. Freudenthal distinguishes between **counting numbers**, by which he means the number word sequence, and **numerosity numbers** which refer to "the potency or cardinality" of a finte set (Freudenthal,1973,170-171). When children grasp the sequential nature of the counting numbers, they become aware of their intrinsic ordinal nature. According to Freudenthal, "In the genesis of the number concept the counting number plays the first and most pregnant role."(p.191) and he criticizes Piaget for ignoring it: "His indifference with regard to the counting aspect is so deeply rooted that he mostly tacitly assumes that his test children can count and he never mentions how far they can count"(p.193).

We tend to agree with Freudental's view that the concept of number emerges from the application of the number word sequence to various enumeration activities. We also agree with Piaget's contention that the concept of order is independent of the number concept, witness the various seriation tasks he has suggested. However, the notion of an ordered set need not be restricted to seriation of physical quantities. In fact, a set can be ordered simply on the basis of the position of its elements. The position of any pencil in a set of ten pencils of different lengths can always be ascertained on the basis of its size. But in a row of ten chips, if the seventh one is removed and the gap it leaves is eliminated by readjusting the row, it will be very hard to re-insert the chip without knowing its precise rank. This example highlights the **ordinal use** of number, that of **measuring the rank** of an object in an ordered set. In this sense, the notion of rank can be viewed as a pre-concept of number.

In our analysis of the notion of rank, we have postulated three distinct components of the child's understanding of this conceptual schema. A first component,which can be considered as an **intuitive understanding** of this concept, reflects a type of thinking based essentially on visual perception. At this level, a child perceives a certain order in a set and can decide about an object coming **before** or **after** or **at the same time** or **together with** another one; whether an object is **between** two other ones can also be determined from a purely visual estimation.

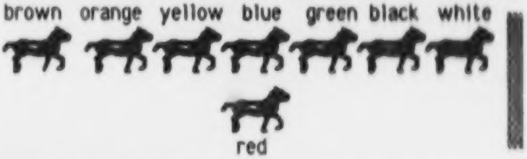
A more advanced level of operation is involved when children can use a more rational procedure to make these judgments about rank and position with reliability and precision. The acquisition of such procedures brings about a deeper grasp of these concepts which can be viewed as **procedural understanding.** These concepts can be assessed by using procedures based on **one-to-one correspondences**. While still being non-numerical in the sense that no enumeration is involved, such procedures can be carried out physically by the children and provide them with an assurance that mere visual estimation cannot achieve.

Still a more advanced level of understanding is evidenced when the child's conception of rank becomes more stable and can resist various surface transformations. The cognitive processes which enable children to overcome the misleading information they obtain from their visual perceptions bring about a level of understanding which we qualify as **abstraction**. It is characterized by their ability to recognize the **invariance of rank** under transformations which change the disposition of the objects without changing their rank.

The present paper describes the different tasks we have designed to assess the kindergartners' knowledge of rank-related concepts. These tasks have been used in semi-standardized interviews with 24 children (average age, 5:8) coming from three different schools in Greater Montreal. The interview dealing with rank lasted about 30 minutes and was videotaped. The same children were interviewed on their knowledge of quantity and their responses are reported in a companion paper, The kindergartners' understanding of discrete quantity by J.C.Bergeron & N.Herscovics.

Intuitive understanding

At the level of intuitive understanding, one can find primitive concepts of rank based purely on visual estimation. The child develops ideas such as **before, after, at the same time or together, between, first and last** without any recourse to numeration. In order to assess this we designed the following task. Eight toy horses of different colors were placed in the row shown below. At first, it was necessary to verify that each child knew the colors we used. Thus the child was given the eight horses and was asked to hand them over to the interviewer who asked for specific colors. These were aligned as shown below:



The questioning proceeded as follows:

(a) **Look, my horses are in a race and here is the finish line.**
Can you show me a horse that is before (in front of) the blue horse?
Are there other horses before (in front of) the blue horse?
(b) **Can you show me a horse which is after (behind) the yellow horse?**
Are there other horses after (behind) the yellow horse?
(c) **Can you show me the first horse? Can you show me the last horse?**
(d) **Can you show me a horse that is between the white horse and the blue horse? Is there another horse between the white horse and the blue horse?**
(e) **Can you show me two horses that come along at the same time (together)?**

Results show that most of the 24 subjects could handle these questions with ease. All children had acquired the general meaning of "before" except one child who interpreted it as "immediately before". Similar results were obtained for the question on "after" where three children had interpreted it as "immediately after". Nineteen of the children understood "at the same time", while five required the expression "together". The notion of "between" was understood by all children who pointed out the two horses between the white one and the blue one. The words "first" and "last" were familiar to all subjects.

As can be seen from the previous tasks, the notion of order and many of its subconcepts exist in the kindergartner's mind. The notion of rank is somewhat more difficult to assess. This is due to the fact that while the child is exposed to all kinds of questions dealing with quantification, those dealing with position in an ordered set are seldom raised. In order to investigate the children's thinking about rank we thought that the notion of **a parade** was quite adapted to our needs since it incorporates the idea of order which is maintained even after motion (which is not the case in a race). A major difficulty we had to overcome was at the level of language.

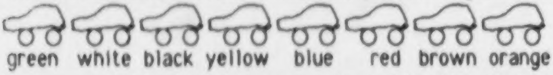
Initially, in our pretests, we had used the word **place** to indicate **rank**. This was understood by some children and not by others. One common misinterpretation was due to the fact that this word is also used to describe the site where an object was, its location. The question **"did it change its place"** could be interpreted in these two ways. Thus while an element in an ordered set might have changed its rank when the first object in the row was removed, some children answered that "its place did not change **because it did not move"**. The same kind of linguistic problems surfaced with the word **"position"**.

Yet every child we had interviewed in our prior research could use the natural numbers in their ordinal sense, that is in their function as a measure of rank. Each subject we had tested in our previous experiments (Bergeron, Herscovics & Bergeron,1986) could identify the second, third, fourth,.... element in a row. Quite interestingly, many children referred to the object's rank as **"its number"** (in French "son numéro"). Thus, we decided that in order to avoid ambiguity, we would use this word and in case it was needed we would convey the meaning we wanted to assign to it, that of rank. The following task was developed to handle the objectives mentioned above.

The subject was asked:

Do you know what a parade is? Have you ever seen a parade? In a parade like this one, the cars follow each other.

A row of 8 little cars, each one of a different color, was aligned in front of the child.



(a) **Can you tell me what is the number of the little blue car?**

(b) **Can you show me the car which is the number seven car?**

If the child did not understand the word "numéro" he or she was asked:

Can you show me the third car?
Can you show me the car which is seventh?
When I say third or seventh, that is its number.
Can you tell me the number of the little blue car?

Of the 24 children tested only 10 interpreted the word "numéro" spontaneously as meaning "rank" and the other 14 were taught. This proportion is somewhat lower than expected but then, in our earlier work, we had interviewed kindergartners five months later in their school year. The word "numéro" provided some minor problem too. In response to the initial question, some children were picking up the blue car and looking for a number which they expected to be inscribed, like on a racing car, but could not find any on our cars. This was due to the fact that "numéro" also refers to "numeral". However with all our subjects, the intended meaning was easily established using the above scheme.

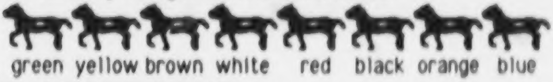
Variability of rank with respect to the quantity of preceding objects

One of our immediate question was to find how well the notion of rank was understood. To

this effect we told the following story:
The parade is now stopped because the green car (the first one) **broke down. The tow truck is coming to get it** (removing the green car).
Do you think that the red car still has the same number as before in the parade?
We referred here specifically to the red car for the child had not used any number to determine its exact rank. The subject thus needed to reason about the question without any specific number in mind. Eighteen of the children thought that the removal of the head car changed the rank of the red car while six did not. We refer to this as the lack of perception of the **variability of rank** with respect to the quantity of preceding objects.

Procedural understanding

As was the case with the notion of quantity, the procedure at stake here was the use of one-to-one correspondence. The tasks were designed to ascertain if these children could use one-to-one correspondences to establish ordered sets in which they had to use the notions of "before", "after", and "at the same time". A row of 8 horses were lined up in front of them and they were given another set of horses:



The children were then told:

I have here some horses on parade.

(a) **Now, can you make a parade in which your red horse comes along at the same time as my black horse?**

(b) **Now I would like you to make another parade in which your red horse comes before my black horse.**

(c) **Now, can you make another parade in which your red horse comes after my black horse?**

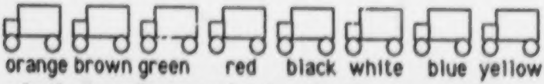
Although we thought these tasks might prove to be difficult, each one of our subjects was able to handle them with ease. They used the interviewer's parade as a template for their own and performed the necessary adjustments to fulfill the constraints that were imposed. These tasks were more difficult than the earlier ones which involved mere recognition of the relative positions. The tasks here necessitated the actual generation of the variously ordered sets.

Abstraction

As mentioned earlier, abstraction refers here to the child's perception of the invariance of rank with respect to surface transformations, that is, changes in configurations which do not affect the rank. Three distinct tasks involving different transformations were designed.

Invariance of rank with respect to the elongation of a row

The first such task assessed the child's perception of the invariance of rank with respect to the elongation of a row. A set of 8 different coloured trucks were laid out in front of the subject:



Look at another parade of trucks. Can you show me the blue truck? Look, the parade moves on (stretch out the row and move all trucks).

(a) **Do you think that the blue truck still has the same number as before in the parade?**

(b) **Do you think that its number is bigger or smaller than before?**

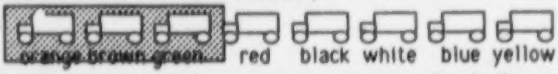
The parade was then stretched a second time and the questions were repeated in order to verify the stability of the answers.

The responses indicate that 19 of the 24 children, (79%), perceived the invariance of rank with respect to elongation. Seven of these, (64%), were among the 12 children under 5:2 and 12, (92%), were among the 13 aged 5:2 or over. Thus, there seems to be a maturation factor involved. The overall success rate here was somewhat higher than in the comparable task on the invariance of quantity (see companion paper) where the rate was 67%, the group of older subjects improving on the rank task, the younger ones having the same success rate on both tasks.

Invariance of rank with respect to visual perception

Our next task dealt with the invariance of rank with respect to the perception of all the units. The row of trucks arranged in the same order as in the last question was laid out in front of the child who was told that the parade would move on and go under a tunnel:

Look, here is a parade of trucks. Can you show me the red truck? Now the parade must get inside a tunnel. (The parade is moved ahead so that the first three trucks are under the tunnel, thus hidden from view):



Do you think that the red truck has kept the same number in the parade? Why do you think so?

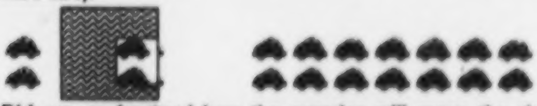
The parade is then moved ahead by another three truck and the same questions are now asked about the blue truck.

The results are most interesting. Fifteen of our 24 children, (63%), thought that the red car had kept its rank even if the three cars preceding it were hidden from view. The second part was aimed at verifying the stability of the initial response. Out of these 15 subjects, 14 still believed that the blue truck had not changed its rank when it reached the entrance of the tunnel. Thus, these responses can be viewed as validated. What is most striking is that while nearly all children failed at perceiving the invariance of quantity when part of a row was hidden, (4% or 13% depending on the task), a majority of these same subjects perceived the invariance of rank when part of the row was out of sight.

Conservation of rank

The following task was designed to verify if the child perceived the invariance of rank in the presence of two rows. The test is similar to Piaget's test on the conservation of quantity. Hence we are calling ours a test on the conservation of rank. The interviewer aligned 9 little identical cars and asked the child to make another parade right next to hers with another identical set of 9 cars. A piece of blue cardboard was set in front of the two parades to represent a river and a small piece cardboard of a different color was used to represent a ferry boat.

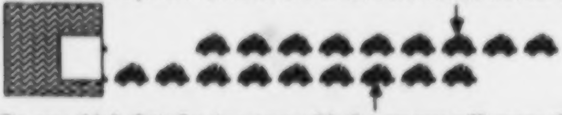
Look, I have a parade of cars which go towards a river. Would you make another parade just like mine? The parades must cross the river in a little ferry boat. But the ferry can only carry two cars at a time, one car from each parade. When the captain is ready he signals for one car from each parade to come on the ferry. (Cross the river with one car from each parade and come back for two more cars):



Did you understand how the parades will cross the river? Good. I'm putting back the four cars in the parades.(After replacing the four cars, the interviewer places an arrow on the 7th car in her parade)

Now I'm putting this little arrow on this car.Can you put this other arrow on the car in your parade which has the same number as mine?

Now look, the parades move on.(Move the child's parade a small distance but move the interviewer's parade further so that in coincides with the fifth car in the other parade)



Do you think that the two cars with the arrows will cross the river at the same time? Do you think the two cars still have the same number?

The results to these questions are quite striking. Only two children out of 24 believed that the two cars would cross the river at the same time. Asked for an explanation, those who could verbalize mentioned that the cars were no longer next to each other. In order to verify that the subjects understood the problem clearly, they were asked to show the interviewer how the parades were to cross the river. Each child demonstrated that he or she had grasped the situation wel by crossing two pairs of cars. After having crossed these two pairs of cars, each child was asked:

And now, do you think that these two cars (indicating the ones with arrows) **will cross together?**

With the two marked cars now in fifth position, only 4 of the children changed their answer . The other 18 held on to their initial view. The children were then asked if the two cars would cross together should the two parades get back next to each other:

If my parade gets back next to yours like before, will the two cars with the arrows cross at the same time?

All children responded affirmatively stating that they would cross together. Their explanation was quite consistent: "the cars would be next to each other". These answers illustrate quite well Piaget's distinction between **reversibility** and **undoing** ("renversibilité"). Our subjects' thinking is not yet reversible in the sense that they cannot as yet **compensate mentally** for the surface transformation they have witnessed. However, they can perfectly well perceive the **undoing** of the transformation which will bring them back to the initial state.

In comparing the results of this task with those of the conservation of quantity, we found that the two children who conserved rank also conserved quantity. But there were eight others who conserved quantity and did not conserve rank. This would imply that from a cognitive viewpoint, the conservation of quantity precedes the conservation of rank, at least in our present culture where experiences dealing with quantity are more frequent than those dealing with rank.

By way of conclusion

As has been shown by these results, the kindergartners' understanding of rank is quite extensive. Their success rate here is remarkable since all our tasks involved the notion of rank in a more abstract form than when related to seriation of physical quantities. Nevertheless, by the age of five and a half, nearly all children can handle order related concepts within the context of the position of the elements of a discrete set. Not only can they all use visual estimation but they in fact can use procedures based on one-to-one correspondence to achieve accurate conclusions. Their perception of the invariance of rank varies with the particular transformations and based on their success rate one can establish the following hierarcy:

Invariance of rank with respect to	N	Percentage
elongation of a row	19	79%
visual perception of objects in a row	15	63%
comparison with translated row (ferryboat)	2	8%

References

Bergeron,A., Herscovics,N., Bergeron,J.C.,(1986), Counting tasks involving some hidden elements, in **Proceedings of the Eighth Annual Meeting of PME-NA**, Lappan,G. & Even,R. (eds.), East Lansing, Michigan: Michigan State University, 21-27

Bergeron,J.C. & Herscovics,N., (1988), The kindergartners' understanding of discrete quantity, **Proceedings of PME-XII**

Brainerd, C.J., (1973), Mathematical and behavioral foundations of number, in **The Journal of General Psychology**, 88, 221-281

Freudenthal,H. , (1973), **Mathematics as an educational task**, Dordrecht-Holland: D.Reidel Publishing.

Piaget,J., (1973), Comments on mathematical education, in **Developments in mathematical education**, Proceedings of the Second International Congress on Mathematical Education, Howson,A.G. (ed), Cambridge University Press

Piaget,J., & Szeminska, A., (1941/1967), **La Genèse du nombre chez l'Enfant**, 4^{e} éd., Neuchatel, Suisse: Delachaux et Niestlé

INITIAL RESEARCH INTO THE UNDERSTANDING OF PERCENTAGES

Rina Hershkowitz and Tirza Halevi, Weizmann Institute of Science, Israel

This paper describes several steps in understanding student behaviour in percentage tasks. The data were obtained from questionnaires and interviews with 6th and 7th graders.
Results show that student responses to types of tasks (which are mathematically similar), are quite different.
The strategies which students used were identified and analysed.
An analysis of patterns of behaviour shows that students also tend to vary their strategies within the same type of task, according to the numbers involved.

I. INTRODUCTION

Percent is one of the most commonly used mathematical concepts in everyday life. However, many students as well as adults lack even an intuitive understanding and cannot use the concept correctly (Hart, 1981, Carpenter et al. 1980,Wiebe, 1986).

The research goals for this project are:

(1) Analysis of student difficulties and thought processes in percentage tasks.

(2) Development of teaching strategies and remedial tools to overcome the above difficulties.

Here we will describe the research conducted to realize the first goal.

There are three types of tasks in percent problems:

i) To find a quantity (A) which is p% of a given quantity (B).

ii) To find what percent (p) one quantity (A) is of another quantity (B).

iii) To find the quantity (B) if we know that (p) percent of it is equal to a quantity (A).

Mathematically the above tasks are all expressed in the one proportion $\frac{A}{B} = \frac{p}{100}$, but some approaches to teaching percent use different strategies for the above three types of task (see, for example, Smart, 1980).

What is the student's "psychological approach" to the different percent tasks?
Does it change from one type of task to the other?
How does student reasoning differ from student to student on the same task?
Do certain number relations encourage certain strategies whether correct or not?

The following is a description of few steps of a study designed to find some answers to the above questions.

II. FIRST STEP

In a preliminary investigation, we administered a questionnaire to students in grades 7 and 8 (N=76) after they had studied percent. The questionnaire included items of the first two types, in two comparative dimensions - accurate computation and estimation. In addition, we conducted unstructured interviews with a few of the students. Students were much more successful with first type than second type tasks in both dimensions (see Table 1).

Accurate Computation			Estimation			
	Find 40% of 150	What percent is 12 of 80?	Estimate whether 53% of 900 is		Estimate whether 60 is ____ of 245:	
Correct	61	26	more than 450	80	more than 25%	59
Incorrect but reasonable	11	18	less than 450	10	less than 25%	29
Incorrect	20	22	450	9	25%	8
No response	8	34	no response	1	no response	4

Table 1: Distribution of student responses (%) to sample tasks of the first two types.

In the accurate computation, most students used a correct algorithm for tasks of the first type, but for those of the second type, if an algorithm was used at all, it was usually different and incorrect.

Many of those who wrote down the correct algorithm in the first type tasks, did not, however, show understanding of the concept.

On the other hand, we found students who showed understanding, but did not necessarily use the standard algorithm.

III. SECOND STEP

In order to understand better the student's conception of percent we decided to investigate mainly the global (intuitive) understanding of the percent concept in the different tasks.

We used a questionnaire in which the students were asked to give their reasoning for each answer, as well as structured interviews.

In most of the tasks, the students were asked to estimate. We believe that estimation reveals intuitive understanding, if it exists.

In order to guarantee "real estimation" (without computation) we used various types of item:

i - Items depicting area or volume without quantification.
ii - Items with "messy" numbers.
iii- Items with a time limit, imposed by the interviewer or by the microcomputer.

We administered the questionnaire to two 7th grade classes at the beginning of the school year. The students had had some formal teaching on the subject in their previous school year.

The answers and reasoning were analysed and followed by recorded interviews with some of the students. In the following, we first describe some of the students' strategies and then various "student behaviour".

Types of Strategy

a) Strategies without any evidence of understanding the concept.

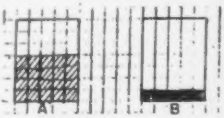
i) Additive strategies level 1.

Here the student adds or subtracts the quantities presented in the task.

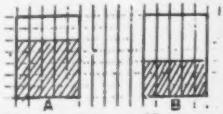
Examples: 1) Hagit (in the interview):

I : You had a collection of 140 match boxes and gave your friend 72 of them. What percent of your collection did you give to your friend? (type 2).
Hagit: About 60 - 70 percent.
I : Why?
Hagit: Because 140 minus 72 is about 60.

2) Hagit (in the questionnaire):
Item 1: "Mark in B 25% of the quantity in A." (type 1).
Hagit shaded the right quantity and "explained":
"Because in A there is 20%, so I added 5%".



Item 2: "The quantity in B is about .....% of the quantity in A." (type 2).
Hagit wrote 40 and "explained":
"I added what we have in A and B and got about 40".



Hagit added the quantities involved, and "named" it percent: When she had squares, she just counted the squares in each quantity, when she did not, she imagined them.

ii) Division strategies level 1.

Here the student divides the given quantities but again no understanding can be identified.

Example: Adi (in the questionnaire): "48% of 150 (type 1), is about 3% of 150 because 48 goes into 150 about 3 times"...

b) Strategies which may reflect some understanding.

i) Additive strategies level 2.

Here the student performs some additive manipulation with the quantities presented, and relates it additively to a "different system" which is somehow supposed to "transform" the result into percentages.

Example: Michal (in the interview)

I : You had a collection of 140 matchboxes and you gave your little sister 120 of them. What percent did you give her? (type 2)
Michal: 80%
I : How?
Michal: I subtracted 120 from 140 and got 20 and then I subtracted it from 100 and got 80%.

Answers that one can get by using this strategy are "reasonable" for an interval of numbers, i.e. when B is "close" to 100 and A < B (when B = 100 we get the right answer). In some cases we had the impression that the student had some global intuitive judgement when (s)he gave quite a reasonable answer, and then when we asked him/her to explain it (s)he created the above algorithm.

ii) Division strategies level 2

These strategies are usually used in type 2 tasks:

In the first one the student checks how many times the smaller quantity goes into the large quantity (B : A).

Example: Naama (in the interview):
I : You had 140 shekel and paid 72 shekel for shoes. Estimate the percent you paid.
Naama: 2% and a little more because 140 : 72 ... 72 goes into 140 about twice...
I : And if you paid 35 shekel, what percent of 140 shekel would that be?
Naama : about 9%
I : When did you pay more, in the first or in the second case?
Naama : In the first, because 72 is more than 35.
I : When did you pay a greater percent of your money?
Naama : (after some hesitation) When I paid 35 shekel ... I think ...

Naama did not feel any conflict in the above situation. But other students used this strategy (B : A) as a first step to the right answer. Example:

Dan : "35 of 140? ... 140 divided by 35 is 4 I think, so it is 25%".

- In the second strategy the student use the inverted division (A : B).

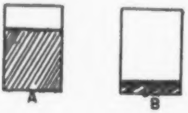
Example: Miri (in the interview): I : Estimate what percent 72 shekel is of 140 shekel?
Miri: 1/2% I : Why?
Miri : Because 72 is about half of 140.
I : Half and 1/2% are the same? Miri: Yes

Miri understands percentages as "part of", but she does not know that it is proportional to 100.

c) Strategies which lead to reasonable answer.

i) Global quantitive judgement.

Here the student uses some wholistic judgement to estimate the relative sizes of the quantities in the given task. It might be that some students use this strategy to check the result obtained by other strategies. But some of them, like Adit in the following example, use only this.

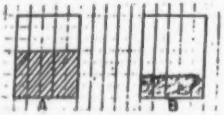


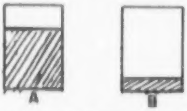
Adit (in the questionnaire): "The quantity in B is about 25% of the quantity in A because in B there is almost nothing and in A there is almost all".

ii) Halving (doubling) and quartering (see Hart 1981).

Examples:

1) Gai (in the interview): "260 of 367 is about 65%, because the difference between 367 and 260 is about 100, so 260 in more than half, therefore it is about 65%".
2) Orly (in the questionnaire):
   Item: "Put in B 25% of the quantity in A". Orly shaded an area in B and explained. "In A we have 50% (she relates the shaded "area" in A to the whole of A), so we must shade half of it to get 25%"
3) Vered (in the questionnaire):
   Item: "The quantity in B is about ....% of the quantity in A". Vered wrote 25% and explained: 4 • 25% = 100%





Hart (1981) notes that: "Doubling and halving are the easiest aspects of ratio, when presented in either problem form or drawing". It is clear that this strategy can be used in only a very limited number of situations. We found that in these situations many students do use it.

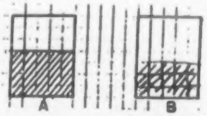
iii) Proportional Strategies

Examples: Michal (in the questionnaire):
Item: "Put 75% of the quantity in A into B". Michal shaded the right area and explained:

$1\% = \frac{1}{100}$

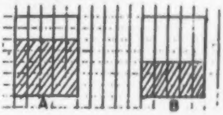
$75\% = \frac{75}{100}$

$\frac{75}{100} = \frac{3}{4}$



Item: "The quantity in B is ....% of the quantity in A". Michal wrote 60% and explained: In A we have 5 rows, in B 3 rows:

$$\frac{3}{5} = \frac{60}{100} = 60\%$$



The above examples are of course evidence of true understanding of the concept.

In the above section we tried to categorise student strategies in first and the second type of tasks, in the hope that it will bring us closer to the understanding of students' percent concept image.

Individual Student Behaviour in Percentage Tasks

Like Hart (1981) in the study on Ratio and proportion, we found that, although some children are very systematic, "most children on interview (and questionnaire) changed the method they used continuously". The change in behaviour seems to be due to the type of task and the numbers involved. Many strategies have some "numerical limitations". Some of these limitations lead (or may lead) to change in student behaviour.

Examples:

1) Gai, in finding what percent A is of B, when A is close to a quarter or half of B, uses halving or quartering, and when the numbers are more "difficult" uses some "difference algorithm" plus quantitative judgement.

2) Miri is usually very systematic. In finding what percent A is of B, she divided A by B when the result is a unit fraction or nearly so; i.e. 10 of 100 is 1/10%, 51 of 100 is about 1/2%, 35 of 140 is about 1/4% etc.... But for 98 of 100 she claimed that she does not know.

- The problem is how to get some overview of students' patterns of behaviour. We have started to use graphical analysis of individual behaviour in order to discern a general pattern. (Wilkening 1979 used it to describe and compare group behaviour).

In type 2 tasks, if one plots a student's answers as a function of the quantity A, with a curve for each value of B, then:

i) If the student uses proportional strategy we will get a proportional graphical model:- The set of curves form a diverging fan of straight lines, the slope of each line is B/100 (see figure 1a).

ii) If the student uses additive strategy we will get an additive graphical model: The set of curves form a parallel fan of straight lines (see figures 1b and 1c).

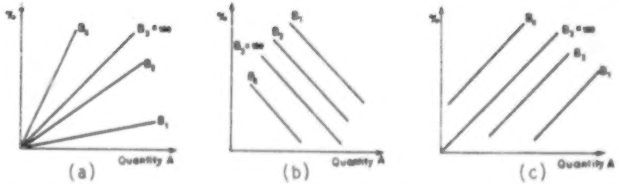


Figure 1: Graphical models of the different strategies:
(a) proportional (b) additive (B-A) (c) additive 100 - (B-A).

It is clear that strategies like inverse division (A ÷ B) and halving also yield the proportional model. Strategies like global intuitive judgement can be either proportional (a) or additive second level (c). We used these models as tools in the graphical analysis of single student behaviour. Examples: In Fig. 2a we see that Hagit for B=140, has changed her strategy from B-A (for A=35, 80) to global judgement (for A=100), and to 100 -(B-A), (for A=120). For B=100 she systematically uses B-A.

Michal uses strategies which lead more or less to the (correct) proportional answer for B=60, 100, 400. But for B=140 (which was the first to be asked) she uses different strategies which are usually wrong. When A is about 50% of B she is very systematic, halving each time

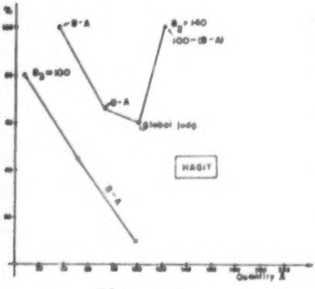


Figure 2a

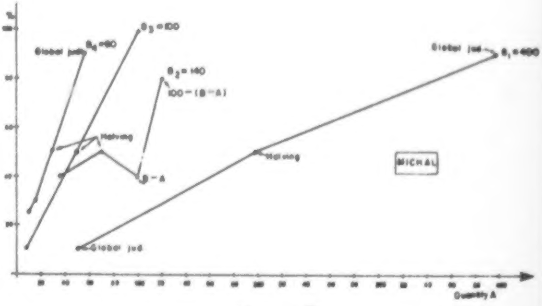


Figure 2b

The above are few steps towards the understanding of individual behaviour in percent tasks. There is more to be done in studying the individual and in studying group behaviour and its quantitative description. By this study, we hope to be able to contribute to the improvement of the teaching and learning the subject.

References

Carpenter, T.P., Corbett, M.K., Kepner JR., H.S., Lindquist, M.M., and Reys, R.E., Results of the second NAEP mathematics assessment: Secondary School. Mathematics Teacher, 1980, 73, 329-338.

Hart, K.M., Ratio and proportion, in K.M. Hart (Ed) Children's Understanding of Mathematics: 11-16. John Murray, 1981, 88-101.

Smart, J.R., The teaching of percent problems. School Science and Mathematics, 1980, 5, 187-192.

Wiebe, J.H., Manipulating percentages. Mathematics Teacher, 1986, 79, 23-26.

Wilkening, F. Combining of stimulus dimensions in children's and adults' judgements of area: An information integration analysis. Development Psychology, 1979, 15, No. 1, 25-33.

ACKNOWLEDGEMENT

We would like to thank Prof. M. Bruckheimer for his contribution to the consolidation of the ideas presented in this paper.

STRUCTURING AND DESTRUCTURING A SOLUTION: AN EXAMPLE OF PROBLEM SOLVING WORK WITH THE COMPUTER[1]

J.Hillel, Concordia University, Montreal
J.-L. Gurtner, Université de Fribourg
C. Kieran, Université du Québec à Montréal

In this paper we analyze a programming solution to a geometric task in which the goal figure is constrained by several conditions.
Our analysis points to an overwhelming tendency on the part of the solvers to proceed by operating on the figure appearing on the screen, rather than on the problem's conditions. Consequently, such problems may end up being 'solved' graphically without an understanding of their embedded relations.

Many types of mathematical problems including numerical, geometric and deductive ones are now given to children to be solved as a computer activity. There are persuasive arguments that the use of computers in problem solving renders it more active, inter-active and engaging.

Certainly, the use of computer encourages an experimental, empirical approach to problem solving. Consequently, children working in a computer environment develop belief systems about what constitutes successful problem solving. Gurtner (1987) discusses some of these beliefs when the problems worked on are of a geometric type. He suggests that one component of the belief system is that success is completely identified with correct-looking screen productions. Thus, 'success' may be achieved even though the intrinsic aspects of the problem are completely circumvented.

It is this last point mentioned above which is the object of our analysis. We reconsider a particular problem solving activity,

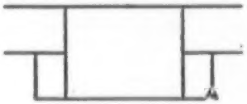
---

[1] Research supported by the Québec Ministry of Education, FCAR Grant #EQ3004.
Dr. J.-L. Gurtner was visiting Concordia University on a Swiss Government FNRS Grant.

already analyzed from a metacognitive perspective by Gurtner, in terms of the relation between the process of solution and the process of understanding.

THE 4-TEE TASK:

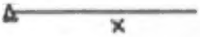
The task was given to six twelve-year olds during Session #4. They were presented with a computer printout



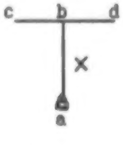
and were asked to write the (Logo) program that would produce the above figure.

The children had at their disposal three Turtle commands for producing the figure. These were:

BASELINE :X in which the turtle 'draws' a horizontal line to its right, X units long, and returns to its initial position.

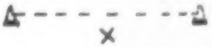


TEE :X in which the turtle 'draws' the figure Tee and returns to its initial position.



ab = cd = X units

MOVE :X in which the turtle moves horizontally X units (to the right if X > 0, to the left if X < 0), without leaving a trace.



With these commands, the goal figure was viewed as four Tees placed on a baseline.

When the figure was presented to the children, some of its features were described as explicit conditions, namely
(i) the small and large Tees were aligned on each side
(ii) the large Tees were contiguous ("no overlap and no gap")
Finally, a constraint on the order of the production of the figure was added:
(iii) the Baseline had to be constructed first (i.e. the program had to begin with the command BASELINE).

Task Analysis:

An exact solution of the task requires that the geometric conditions (i) and (ii) above be reinterpreted as numerical relations which govern the choice of inputs to the commands BASELINE, TEE and MOVE. Thus, labelling parts of the figure as follows

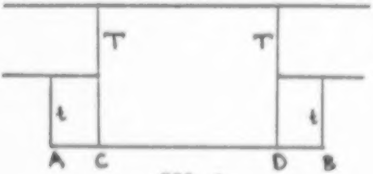


FIG. 1

and letting t and T correspond to the inputs for the small and large Tees, we have the following length relations:

$AC = DB = 1/2\ t$

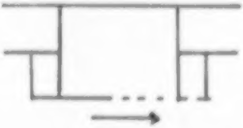
$T = 2t$ (alignment condition)

$T = CD$ (contiguity condition)

These relations establish an implicit relation between the length AB (which is the input to BASELINE, the first command in the program) and the length t (which is the input to TEE, the second command in the program). Finding the actual relation between t and AB is non-trivial and its derivation requires several algebraic substitutions, i.e.

$AB = AC + CD + DB = 1/2\ t + T + 1/2\ t = 1/2\ t + 2t + 1/2\ t = 3t$.

Now, without our imposition of condition (iii), it would not have been a very difficult task for these children. They would have, in all likelihood, constructed the four Tees first and then fitted the Baseline by a sequence of visually-based trial and error adjustments, i.e.



By adding the constraint that the Baseline had to be chosen first, we greatly increased the complexity of the task. It meant that, having arrived at an incorrect solution, the children would have to (i) identify the appropriate input to be adjusted, (ii) having change this input, and (iii) to reestablish all the relations with the other inputs. In particular, trial and adjustment strategy could not proceed by isolating and modifying a single input.

Two aspects of the children's solution interest us here:
(a) Understanding the problem and, in particular, the realization that, once having chosen a fixed Baseline, all the other inputs were determined. We did not expect that the children would be able to link t to AB (the unobvious relation t = 1/3 AB merely assured us that the problem would not be solved surreptitiously). We did expect that the children would eventually realize that t was the only input which they could freely modify, if they had opted for a trial and adjustment strategy
(b) The choice of inputs and, specifically, whether the inputs satisfy one or several explicitly derived relations.

We proceed by analyzing the solution process of one child, which was rather typical.

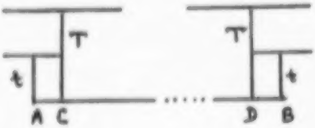
ROSA'S SOLUTION:

Rosa had already spent most of the previous session (session #4) on the 4-TEE task. In session #5 she restarted it, without looking back at her previous attempt.

Initial solution of session #5:

Rose's initial program, which was similar in nature to the one she had produced in the previous session, was strongly influenced by the symmetry of the figure - whatever was done on the left side of the Baseline had to be done on the right side as well. Her program had the following structure:

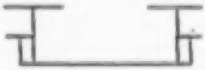
```
BASELINE AB
TEE t    }
MOVE AC  }  left side
TEE T    }
MOVE CB
TEE t    }
MOVE BD  }  right side
TEE T    }
```



Her inputs were AB = 130, t = 20, T = 40, AC = BD = 10 and CB = 120.

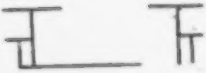
We note that her initial attempt was very controlled. The choice of inputs was done with care, and the relations T = 2t, AC = 1/2 t and CB = AB - AC were all satisfied. At this point there was no particular linking of t to AB, except in that AB was quite large compared to t and T. This might have been a deliberate strategy on her part so as to allow her more manoeuvrability; in the previous session, she had consistently chosen t as 1/2 AB which resulted in a large overlap of the big Tees.

Output i



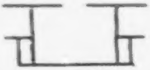
Rose's spontaneous reaction to the output was to 'shrink' the Baseline. AB was decreased from 130 to 90 but none of the other inputs was touched. Her expectation was that this action would have an 'accordion' effect of bringing the Tees on the left and right closer together, as if she were dealing with a rigid figure.

Output 2



The output gave Rosa a clear indication of what else had to be modified. She then reestablished the relation CB = AB - AC by decreasing CB from 120 to 80.

Output 3

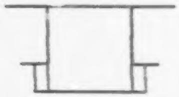


The output indicated that the the goal figure was now within reach. The large Tees were much closer to each other than before and this suggested an obvious action for closing the gap, namely, to operate on those Tees. This is, in fact, what Rosa did. She started to close the gap by a sequence of stretches of these Tees. Thus the input T, initially 40, underwent six cautious increases, each of which was followed by an output on the screen. When T was set at 63, no gap appeared in the output which led Rosa to conclude that the large Tees were now contiguous.

Rosa's actions were the start of the 'destructuralization' of the solution. In her effort to close the gap, she forgot that T and t were linked by the relation T = 2t and that T should not be changed on its own. She was so preoccupied with closing the gap that she didn't even notice in the outputs that the small and large Tees were no longer aligned.

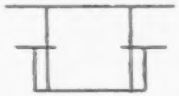
Furthermore, in contrast to her way of choosing inputs earlier, the new values of T were not based on any explicit relations (such as T = CD). Rather, she adopted what we have termed the 'qualitative' approach of 'making bigger' (see Kieran et al.,1987). In fact, the large Tees were now overlapping, something that was not discernable by looking at the output. Rosa had, in fact, replaced a gap by an overlap and lost the alignment condition in the process.

Output 4



Rosa expected the output to indicate a successful solution. However, now she did notice that the small Tees were too small and not aligned with the large Tees. She continued the destructuralization of her initial solution by ignoring the relations T = 2t and AC = 1/2 t and proceeded with a single stretch, changing the input t from 20 to 28, leaving all other inputs in the program unchanged. Her Tees now were neither aligned, nor contiguous nor correctly placed on the Baseline.

Output 5



Rosa realized that she was not getting any closer to a solution and gave up on the task.

DISCUSSION:

There are certain features of the attempted solution by Rosa which were quite prototypical of the way most of the other children solved this and similar problems. Her initial solution, planned away from the computer, respects most of the relations governing the lengths of the different components of the figure. However, as the solution process progresses, the screen output becomes the relevant 'data'. There is no longer any attempt to either satisfy already established relations or derive new ones from the given conditions. Qualitative and local solution strategies become dominant; an initially structured solution becomes progressively more destructured and ad hoc.

This solution behaviour was prevalent even among children who ended with a 'successful' solution (in the sense that the output on the screen seemed to satisfy the required conditions). They might have persisted longer with 'patching up' the different outputs or, eventually, adjusted the Baseline to fit the Tees, thus ignoring one of the explicit constraints. In either case, they were no closer to really understanding the nature of the problem.

Most research into problem solving has pointed to a frequent alternation, while solving a problem, between the solution phase and the understanding phase. To quote Simon (1978), "The solving process appears to exercise overall control in the sense that it begins to run as soon as enough information has been generated about the problem space to permit it to do anything. When it runs out of things to do, it calls the understanding process back to generate more specifications of the problem space" (our emphasis). To the extent that the above typifies problem solving behaviour, the behaviour that we have described seems rather anomalous. We put forward the following explanation for this: Using a computer is an action-oriented activity; once a solution phase is started one seldom runs out of things to do. Consequently, the process of understanding may simply not have an occasion to be called upon.

BIBLIOGRAPHY:

Gurtner, J.-L.,(1987). Success and understanding while solving geometric problems in Logo, Proceedings of the Eleventh International Conference of the Psychology of Mathematics Education (Vol.II, pp 229-234), Montreal.

Kieran, C., Hillel, J. and Gurtner, J.-L., (1987). Qualitative strategies in Logo centering tasks, Proceedings of the Third International Conference for Logo and Mathematics Education (pp 126-138), Montreal.

Simon, H.A., (1978). Information-processing theory of human problem solving, in W.K. Estes (Ed.), Handbook of Learning and Cognitive Process, Vol.5, J. Wiley & Sons (New York).

METACOGNITION: THE ROLE OF THE "INNER TEACHER"(3)

Ichiei HIRABAYASHI and Keiichi SHIGEMATSU
(NARA University of Education, Japan)

ABSTRACT

The nature of metacognition and its implication to mathematics education is our ultimate concern to investigate through a series of our researches. We argued in the last two papers that metacognition is given by another self or ego which is a substitute of one's teacher and we referred to it as "inner teacher". In this paper we will show a more concrete description of pupils' metacognition through teacher's responses of the questionnaire. Especially we will prove that there is a close correlation between pupils' metacognition and teacher's utterances in class sessions.

AIMES AND THEORETICAL FRAMEWORK OF THE RESEARCH

The ultimate aim of our research is to have the clear conceptions about the nature of metacognition and to apply this knowledge to improve the method of teaching mathematics. This paper is the report on the preparatory works for this aim.

In our former papers presented to PME annual conferences, we have argued that metacognition would be formed through teachers' behaviors and utterances in the classroom lessons. If we may use a metaphor, teacher enters in the pupil's mind through the lesson and becomes another self of the pupil, monitoring, evaluating the original self's activities. So we have referred to this another self as *inner teacher* because it plays the same role as the actual teacher in the teaching-learning situation.

The favor of this metaphor is that we could have the practical methodology to investigate the nature of metacognition; that is, we may only collect many varieties of teachers' behaviors and utterances in the lesson and carefully examine and classify them from some psychological view-points.

Following to this research scheme, we have done two works:

1) we have collected teachers' utterances through the lesson observation and make the list of the questionnaire both to teachers and students to know which items are the most used during lessons by teachers. Then, we have compared the two kind of these responses, one is from teachers and the other is from students.

We think that the items common to the both responses would be suggest the essential components of metacognition. We should add to say that the data from students were from university students of mathematics course for elementary school teacher and nonmathematics course for the same, and the contrast of these two kind of students seemed to suggest some important things about the nature of metacognition.

2) we have classified the said list of teachers' utterances for the lesson of the problem-solving situation. As we will show later this situation is the most promising to investigate metacognition and we had also here some interesting results suggestive to our future direction of the research.

METHODOLOGY OF THE RESEARCH

1. Teacher's Utterances in Class Sessions

(1) Making the list of questionnaire

We have gathered teachers' utterances from the recorded teaching-learning processes. On these records, we made the list of questionnaire. We classified these items of questionnaire into 4 classes according to the types of teachers' behaviors in the lesson:

1) explanation 2) question 3) indication 4) evaluation

From each category, some items are shown in the following;

1) explanation
   "If you can draw a figure, you may solve problem."
   "I(teacher) myself used to make a mistake."
2) question
   "Can you use this strategy at any place?"
   "Can you explain the reason for it?"
3) indication
   "Read the problem carefully."
   "Please give me an example for that."
4) evaluation
   "Good!"
   "You could have grasped the important point."

(2) Data collection

We sent the questionnaire to teachers in all levels of school and had responses from them, numbers of which were as follows:

1) Elementary school teacher	38
2) Junior high school teacher	24
3) High school teacher	16

2. Students' Impressions about Teachers' Utterances

We have used the same questionnaire to analyse university--students' impression of their teachers' utterances in their school days. This is because, as we argued, teachers' utterances would have became the important components of students' metacognition.

We collected the data not only from students in mathematics major, but also in non-mathematics major. The numbers of each were as follows:

1) Student of mathematics major	29
2) Student of non-mathematics major	44

3. Metacognitive Framework of Problem-solving

A classroom lesson includes varieties of activities of students and among them we notice the so-called problem solving activities are the most preferable phenomena to think over the nature of metacognition, because there we may observe many features of this complicated concept. Thus, we exclusively concerned with these learning situations in our research of metacognition.

At first we introduce the classification framework of teachers' utterances, which has two dimensions: one may be referred as the *problem solving stages* and the other as *metaknowledge categories*, and so we have 24 sections in all as is shown in the following figure. The former dimension is suggested from that of Schoenfeld and the second from that of Flavell and both of them were a little modified by us.

(Figure 1) Metacognitive framework in problem solving

1. GENERAL STAGE
    11) environment 12) task 13) self 14) strategy
2. ANALYSIS STAGE
    21) environment 22) task 23) self 24) strategy

3. DESIGN STAGE
    31) environment 32) task 33) self 34) strategy
4. EXPLORATION STAGE
    41) environment 42) task 43) self 44) strategy
5. IMPLEMENTATION STAGE
    51) environment 52) task 53) self 54) strategy
6. VERIFICATION STAGE
    61) environment 62) task 63) self 64) strategy

Some comments will be needed about this framework.

To the Schoenfeld's stages we add the 'general stage' in the beginning, because we think that there are some metacognitions which can not belong to the specific stage of him but have influences to all stages; for instance,

"Don't be afraid of mistake, you may do mistake."

would be made in any stage of students activities.

RESULTS AND DISCUSSION

1. Categorization of items

Contrasting responses from teachers and students, we classified them into three categories according to the frequency of coincidence, as follows;

1) Category I

In this category each item is responded by above 50% of the teachers and above 50% of the students. Some examples are as follows:

"Do you have any question?"
"Try to figure it out by yourselves."
"Yes, sure!"

2) Category II

In this category each item is responded by above 50% of the teachers but by only a few students. Some examples are as follows:

"You already experience in solving problem similar to this."
"What is the given condition?"
"If you can solve problem by a strategy, try to solve it by another strategy."
"It is an interesting strategy."

3) Category III

In this category each item is responded by only a few teachers but above 50% of the students. Some examples are as follows:

"This is a good problem"

"How can you describe it in the expression?

2. Some different utterances according to the school level

There are some difference in the number of responses according to the school level.

1) Elementary school teacher

"What is the given condition?"

"Solve the problem in any way you like."

"You are bright."

2) Junior secondary school teacher

"If you can draw the figure, you can solve the problem."

"When you have finished, please check the problem and your answer once more."

3) Senior secondary school teacher

"Have you finished?"

"If you lost your way in solving the problem, please read and analyse the problem once more."

3. Teacher's Utterances in the Problem Solving

Here we mention some interesting utterances that might have some connections with the formation of metaknowledges in each stage of problem solving situation. Some items are as follows:

1) general stage

11) "You may make mistakes."

12) "This is the first time for you to solve this type of problem."

13) "Solve the problem carefully."

14) "Solve the problem by yourselves without other's help if possible."

2) analysis stage

22) "You have already the experience in solving problem similar to this."

24) "If you can draw the figure, you can solve the problem."

3) design stage
  34) "This problem may not be solved by computation only."
4) exploration stage
  44) "Try to reduce the problem to a easier, and similar problem."
5) implementation stage
  52) "This problem may be slightly difficult from the previous ones."
  53) "Don't do too many things at a time, or you may mistake."
  54) "How can you describe it in the expression?"
6) verification stage
  62) "This problem is interesting."
  63) "If you can't understand the problem and don't know the answer, you must review it once more."
  64) "Can you use that strategy at any time needed?"

In some sections of this framework, we can't find teacher's utterances from this questionnaire.

CONCLUSION

1) In the classification of teacher's utterances, we can clearly notice that teachers speak very often for 'indication' to children. This may mean that in our country teachers are apt to assume an attitude to 'teach' not to make pupils learn of their own accords.

2) In the framework of problem-solving (figure 1), we see that few utterances belong to sections 12), 14), 44) and 63). This may show that teachers often emphasize the strategy of solving exclusively, taking less care of other important features of solving activities.

3) In the comparison between data of teachers' and students', we can guess that teachers speak not so much in the stage of 'design' and 'exploration', but students have received much impression from teachers' utterances of these stages.

4) The comparison between students of mathematics major and non-mathematics major in university shows that the former may have much metaknowledges concerning to the positive attitude toward problem-solving, while the latter seems to stick too hard to stages of analysis and implementation.

5) Teachers' utterances are different according to the kind of school level: Elementary school teachers' utterances cover all of stages of problem solving, but teachers of higher levels incline only to

speak more in the particular stages of problem-solving especially of 'analysis' and 'strategy'.

In this report we think that we could have clarified in some degree the close relation which the teachers' utterances has to the formation of metacognition of the students, but we are still very far from analysing the mechanism of the formation. Through we personally believe that there would be the critical period of this formation in around 3rd grade in the elementary school, the verification of this fact must be left to our future researches.

Finally we should thank to Prof. F.K.Lester,Jr. and Prof. J.Garofalo for having much instructions from their works. We think our research is different from theirs in the next two points:

(1) They seem to have their data through the individual teaching and interviews, but our data originates from the daily classroom lessons.

(2) Their data seem to come mainly from high schools, while ours cover all levels of schools.

REFERENCES

1. I.HIRABAYASHI & K.SHIGEMATSU: *Meta-cognition: the Role of the "Inner Teacher"*, PME10, 1986.7, (165-170).
2. I.HIRABAYASHI & K.SHIGEMATSU: *Metacognition: the Role of the "Inner Teacher"(2)*, PME11, 1987.7, (vol.2,243-249).
3. F.K.LESTER,JR. & J.GAROFALO: *An Emerging Study of Sixth-grader's metacognition and Mathematical Performance* , the annual meeting of the American Educational Research Association,1986.4.
4. F.K.LESTER,JR. & J.GAROFALO: *The Influence of Affects, Beliefs, and Metacognition on Problem solving Behavior: Some Tentative Speculations*, the annual meeting of the American Educational Research Association,1987.4.

FORMALISING INTUITIVE DESCRIPTIONS IN A PARALLELOGRAM LOGO MICROWORLD

Celia Hoyles & Richard Noss
Institute of Education, University of London

Abstract: This paper reports a follow-up study to that presented in Montreal at PMEXI, (Hoyles & Noss 1987) in which we reported on an investigation of pupils' interactions in a Logo-based parallelogram microworld. In this study, we take account of pupils' initial and final conceptions, and present findings on how understandings developed in the computer context were synthesised with those developed within other domains.

The framework within which this study was located consists of four dynamically related components of mathematical understanding: the *use*, *discrimination*, *generalisation*, and *synthesis* of mathematical notions (UDGS). Such a model of learning presupposes an environment which allows pupils actively to construct their own understandings on the basis of informative feedback. An interactive computer environment can (under appropriate experimental conditions) fulfill such a role.

In this earlier study we noted some confusions between turtle turn and angle. We also found that pupils frequently constructed procedures with more than one variable (input), and used them without making the relationship between the variables explicit within the program -- these we referred to as implicit, action-based generalisations -- and we noted that an awareness of the relationship at a conscious level would be unlikely to occur without intervention. The study also identified different levels of discrimination: discrimination of the features of the figure without regard to its available symbolic representation, and discrimination within the symbolic representation without regard to its visual outcome. Finally, we observed how the symbolic representation of a computer program acted as a form of *scaffolding*, (Hoyles and Noss, 1988) allowing the pupils to sketch out their global structuring of the problem before turning their attention to local detail. An overall conclusion concerned the importance of pupils' coming to synthesise the symbolic description with the geometrical image.

Subsequent to the study we noted the need to investigate the following points:-

- pupils' conceptions of the relevant mathematical notions prior to the experimental phase;
- if and how understandings developed in the computer context were synthesised with those developed within other domains.

We had also hoped to probe pupils' classification of squares or rectangles in terms of the set/subset relationship to parallelograms, but in the event were unable to do so. These three issues constituted the objectives of the present research.

Methodology

We undertook a study with six 13-year old Logo-experienced pupils. Our experimental methodology consisted of the following research instruments all of which were piloted and appropriately modified prior to the main study:-

- a pre-test consisting of an audio-recorded semi-structured interview, including some written responses, to probe pupils' conceptions of parallelograms, rectangles and squares;
- a structured set of Logo based tasks, some to be attempted on the computer and some off the computer;
- a post-test, again consisting of an audio-recorded semi-structured interview, including written responses, to investigate what pupils had taken away from the experimental work, and in particular whether there were any changes in their conception of parallelograms, squares, rectangles etc.

The pre-test, which was administered on the day preceding the structured tasks, sought to investigate:-

- how pupils spontaneously described a parallelogram; how they would draw one and write down a definition;
- whether pupils were able to recognise correctly instances and non-instances of parallelograms in a set of 13 shapes (including rectangles, rhombuses and squares, as well as irregular quadrilaterals), and how they would justify their decisions -- including convincing another pupil;
- whether pupils would be able to construct a procedural description of a parallelogram in a 'real-world' context (of walking around a path) and in the form of a Logo program;

The structured tasks followed a similar pattern to those in our previous work -- with specific questions to be answered on and off the computer -- but with some modifications. The pupils were given a Logo procedure for a parallelogram, SHAPE, with the turns of the parallelogram (rather than the lengths of the sides as previously) parameterised as follows:-

```
TO SHAPE :ANGLE1 :ANGLE2
    FD 200 RT :ANGLE1 FD 100 RT :ANGLE2
    FD 200 RT :ANGLE1 FD 100 RT :ANGLE2
END
```

They were then asked to:

- predict the screen outcome of typing SHAPE 30 150;
- construct a tiling pattern on the computer using SHAPE;
- draw seven different parallelograms (all with sides 200 and 100) in different orientations (including rectangles and squares) using their SHAPE

procedure -- rather than leaving, as we had previously done, the choice of construction method to the pupil;

- modify the SHAPE procedure to a procedure with *one* angle input only (called NEWSHAPE). This aimed to see if they were aware of any necessary relationship between ANGLE1 and ANGLE2, (i.e. that their sum must equal 180°) and, if they were, whether they could make the relationship explicit in the procedure;
- construct a procedure which would draw *any* parallelogram, no matter what size or shape (called SUPERSHAPE). Such a procedure would in fact need *three* inputs. In order for the pupils to reflect upon the generality of their SUPERSHAPE procedure, we built in a communication aspect to the task: each pupil was asked to draw any parallelogram he or she liked, label its sides and angles, and give it to another pupil who would then try to draw the parallelogram with his or her version of SUPERSHAPE: the final outcome to be discussed by the two pupils.

The structured tasks were undertaken during a whole-day session in the University computing laboratory. Data was obtained using 'dribble files' of the pupils' work, the researchers' notes, and the written work of the pupils.

The post-test was administered immediately following the structured tasks and was designed to probe pupils' conceptions of the Logo procedures for parallelograms they had constructed, whether the understandings they had developed during the tasks had affected their view of the nature of parallelograms and, in particular, their (possibly new) classification of rectangles, squares and rhombuses with respect to the set/subset relationship with parallelograms. The pupils were given the following procedure:

```
TO SUPERSHAPE :SIDE1 :SIDE2 :ANGLE
   FD :SIDE1 RT :ANGLE FD :SIDE2 RT 180 - :ANGLE
   FD :SIDE1 RT :ANGLE FD :SIDE2 RT 180 - :ANGLE
END
```

They were:-

- asked to describe what shapes SUPERSHAPE would draw with different inputs, justify their descriptions and draw, in particular, what SUPERSHAPE 100 240 would produce;
- asked if and how SUPERSHAPE could draw rectangles, squares and rhombuses;
- given exactly the same recognition task as in the pre-test; that is, asked to pick out instances and non-instances of parallelograms in a set of 13 shapes, giving reasons for their choices;
- finally, asked whether *all* the instances of parallelograms in the recognition task could be drawn with SUPERSHAPE, and whether they could use SUPERSHAPE to draw shapes that were not parallelograms.

Findings

We concentrate on three areas of interest which emerged from analysis of the data: the ways in which the pupils defined a parallelogram and how this definition interacted with their activity, the relationship between visual and symbolic representations, and the pupils' initial and final conceptions of the relationship between set and subset.

Pupils' definitions of a parallelogram

In the questions on the pre-test designed to investigate spontaneous descriptions of a parallelogram, all the pupils drew what can be termed a prototype parallelogram; that is, a parallelogram with a pair of horizontal sides usually leaning to the right. Their definitions of a parallelogram were all declarative, based on the equality of sides and angles. There was, however, an assumption that a parallelogram had to be 'slanted'. This was either stated explicitly in the definition of a parallelogram: for example Gail wrote 'The opposit (sic) sides and angles are equal. It is slanting'. Alternatively, it emerged later during the recognition task, when a rectangle was rejected as an instance of a parallelogram: for example, Lyndsey stated 'The angles are not meant to be 90 -- a parallelogram is a twisted square or rectangle....it's meant to be squashed'.

This throws light on pupils' perception of definitions and their ability to use them -- and in particular, the frequent mismatch between pupils' formal definitions and their intuitions. Thus Lyndsey's formal definition was 'all sides are equal, opposite angles are equal', yet her intuitive definition was 'it's either a rectangle or a square squashed'. Similarly, Adam knew at a formal level that parallelograms have two equal and opposite sides and angles, but excluded rectangles and squares which had sides which were horizontal/vertical. He decided however, that the square that was tilted over *was* a parallelogram, presumably because it displayed 'slantiness'. Matthew was more aware of this confusion and refused to answer whether squares and rectangles were parallelograms -- saying "They're not parallelograms because of the right angles. But I'm not sure (it looks like one)!".

The relationship between visual and symbolic representations

a) Concerning explicit geometric attributes: When asked to draw the figure (away from the computer) that would be produced for SHAPE 30 150, all the pupils drew a parallelogram, although there was some confusion in the labelling of the angles (similar to that reported in the previous study) and in the orientation of the shape. In justifying why a parallelogram was the outcome, the answers made general references to a parallelogram's properties (for example opposite sides being equal) without any explicit reference to the features of the code relating to these geometric properties. Thus there was, at this stage, little evidence of synthesis of the visual and symbolic. For example, Gail drew and labelled her parallelogram correctly, but when asked to write a procedure which would produce a given shape, she inserted the inputs the wrong way round.

We noted in the post-test a tendency towards a more precise definition of relationships; for example, while pretest definitions tended to involve 'slanting' or 'squashed' squares, post-test responses focused rather more on the features of the parallelogram which had been explicitly discriminated during the activity (such as the equality of alternate turns in SUPERSHAPE). We noted an increased readiness to discriminate at the symbolic level, rather than only the visual. For example, on the post-test (but not on the pretest), Lyndsey and Simon both pointed to the code to justify their responses.

b) Concerning implicit geometric relationships: Despite the confusions exhibited in the pretest over whether rectangles were parallelograms or not, all the pupils found no problem in using SHAPE correctly to draw rectangles (i.e. by using 90 90 as inputs). Additionally, all the pupils were successful in drawing the seven parallelograms with appropriate inputs to SHAPE (i.e. inputs whose sum was 180). However, when subsequently they were asked to construct NEWSHAPE with only *one* angle input, their lack of awareness of the relationship was very apparent. Lyndsey, for example, was completely baffled: when challenged to explain how she had obtained the correct inputs in the previous question, she replied: "I took the angle and doubled it, subtracted from 360, and halved it to get the other input". She could not convert her complicated procedure for calculation into a formal relationship which could be used in NEWSHAPE. In contrast, Gail used the same calculating procedure but did manage to formalise it by writing on paper, RT (360 - :ANGLE * 2)/2 which she then 'tidied up' to RT 180 - :ANGLE on the computer.

In fact both these girls and a third -- Emma -- used two pieces of information about parallelograms which they considered as flowing from their definition (i.e. the sum of the angles was 360, and opposite angles were equal). They were so busy doing these calculations -- which worked, of course -- that they did not reflect on the values of the two inputs or see the *simple* relationship between them. An intervention was required at this stage, merely to provoke the pupils to take another look:

Researcher:"Can you see any connection between the inputs to SHAPE?"
Lyndsey: (immediately) "Oh ... they add up to 180."

However, the understanding generated by this intervention turned out to be only transitory. Lyndsey subsequently wrote:

```
TO NEWSHAPE :ANGLE
   FD 200 RT :ANGLE FD 100 RT :ANGLE - 180
   FD 200 RT :ANGLE FD 100 RT :ANGLE - 180
END
```

Thus she made a common error in converting 'they add up to 180' into mathematical language. However, when she tried out NEWSHAPE on the computer, it did not produce a parallelogram. She then debugged her procedure *visually* -- i.e. she saw that she should type LT rather than RT after the FD 100 command, and produced the following workable procedure:

```
TO NEWSHAPE :ANGLE
    FD 200 RT :ANGLE FD 100 LT :ANGLE - 180
    FD 200 RT :ANGLE FD 100 LT :ANGLE - 180
END
```

Thus there was no ultimate synthesis between the ordinary language to describe the relationship between the two inputs, the Logo code and the visual outcome -- on this occasion the computer allowed her to circumvent an explicit *symbolic* generalisation. Lyndsey had not *really* grasped the geometric relationship, as was evident in her post-test where she again used her previous calculation to find the second turn. Similarly Gail, despite deriving the relationship correctly within NEWSHAPE, seemed to lose sight of it when she came to *use* NEWSHAPE in subsequent work -- trying inputs of 70 and then 120 to create a parallelogram whose first internal angle was 70°. This data throws light on the cyclical nature of the UDGS model we have proposed elsewhere (Hoyles & Noss 1988) concerning the way in which, during the *use* of a procedure which has first been constructed, attention shifts *away* from the symbolic and towards the visual. Thus the symbolic relationship was made explicit during the construction of NEWSHAPE, but when the procedure became a tool, the *consequences* of this relationship were ignored.

The work of these three girls contrasted with Adam. The girls all worked in direct mode on the computer, stamping the procedure on the screen, typing ar interface and stamping another procedure. Adam (and the other two boys) worked all the time in the editor. He constructed NEWSHAPE correctly, *but* chose the wrong size of input for the shapes required -- he always chose the complementary input in NEWSHAPE -- e.g. 30 when NEWSHAPE 150 was required. Thus he focussed on the symbolic code of his programme, and had not integrated its components and sequence with effects on visual outcome. This highlights a further difficulty in switching from the computer to pencil-and-paper -- the latter really had no real payoff for the pupil.

Overall, there was therefore evidence of synthesis between the visual and symbolic representations at the level of *definition* of a parallelogram -- that is, how the geometric attributes of the parallelogram in terms of equality of opposite sides and angles were reflected in the Logo code; but *not* at the level of geometric relationships inherent within the *construction* of the parallelogram.

The relationship between subset and set

One way in which we were able to gain insight into the way in which the idea of parallelogram was conceived, was by probing the extent to which pupils viewed special cases such as rhombuses, rectangles and squares (in various orientations).

a) Rectangles and squares: As we mentioned above, all the pupils in the pre-test were confused as to whether or not rectangles and squares were parallelograms. Despite this, none of them found any problem with immediately using SHAPE for producing rectangles. Thus they were prepared to see that the general procedure SHAPE would produce rectangles *as special cases* (when the inputs were both 90), even though they did not acknowledge rectangles as instances of parallelograms. After the experimental phase, five out of the six pupils were willing to see rectangles as parallelograms i.e. they were willing to reject -- albeit tentatively -- their intuitive ideas and those features of their prototype parallelogram which were not necessary

For example, with reference to a rectangle, Lyndsey said: "It is because opposite angles are the same and opposite sides are the same, and that is what a parallelogram is. Before I said a parallelogram is not a square or a rectangle. I still see that is sort of right, but now I see it doesn't *have* to be squashed." Simon would not commit himself: "Well it can but -- I can't -- I don't -- I don't actually think it's a parallelogram. It can be if -- working it out the way that you do on the computer. It's like er... I'll put in various angles for the SUPERSHAPE -- so it can make one of those (i.e. a rectangle) out of a parallelogram....But I don't actually think it's a parallelogram....I think it's a rectangle."

b) Rhombuses: In the pre-test recognition task all the pupils identified the rhombus as an instance of a parallelogram. In the experimental phase they drew rhombuses correctly, but in the post-test five of them gave 90 as the *only* possibility for the angle input to SUPERSHAPE (while the inputs to the two sides were correctly given equal values). It was apparent that they were unclear as to the variants and invariants of the rhombus's geometric attributes. Interestingly enough, it seemed that they thought rhombuses had turns of 90°, yet did *not* refuse to designate a rhombus as a parallelogram -- which contrasted with their professed intuitive definitions (which explicitly excluded right angles). We conjecture that the focus here was on the lengths, not the angles. Adam was the only exception: he had a precise definition of a rhombus which related it specifically to a parallelogram as well as a square: "A rhombus is a square parallelogram".

Conclusions

We are able to conclude that the mismatch between the pupils' fuzzy and intuitive ideas of a parallelogram and their formalised definitions identified in the pretest, was at least partially resolved as a result of participation in the experiment: we conjecture that *using* the formal code helped to discriminate the significant features of a parallelogram. As far as the relationship between the turns is concerned, the pupils were able to make it explicit when requested, but it is far from clear how far they saw the functionality of the generalisation thus gained, or were able to keep the relationship in mind when the procedure was used as a module in a larger project.

A related aspect is the clarification of the set/subset relationship. There was some evidence that the experience of confronting the relationship between parallelogram and rectangles (by using and generalising the given procedures) did have the effect of introducing uncertainty into some of the pupils' conceptions. We hypothesise that the initial confusion displayed by the pupils might arise from the fact that pupils intuitively know that {apples} is a subset of {fruit}. This is different from the situation involving rectangles and parallelograms: a slightly deformed rectangle is very much like a rectangle -- and not far off being a typical parallelogram. The important relationships change from those *between* the angles to the *size* of the angle. As far as rhombuses were concerned, pupils initially saw them as tilted squares and defined them as parallelograms in contrast to horizontally oriented squares, which were excluded. We conjecture that in this case, the essential intuitive feature of parallelograms -- their 'slantiness' -- was crucial. This initial conception of rhombuses persisted in the post-test.

We conclude by making three further points. Firstly, we found that our interpretations were handicapped by not having the backup of longitudinal data, and indeed not having a close relationship with the children (this situation was quite different from that in our earlier work). Secondly, we noticed that the rather directed nature of the tasks resulted in: i. some differences in approach from other studies we have undertaken, (for example, we noted very few instances of pupils using the computer as scaffolding presumably because insufficient scope was allowed for experimentation) and ii. the danger that pupils almost inevitably produce a result, but without necessarily understanding how their actions led to this result. Thirdly, 5/6 pupils in the post-test, in answering a question in which they are asked to pick a shape which they know is a parallelogram and write down a Logo procedure for it, wrote a procedure in direct mode. We interpret this finding as suggesting that the idea of SUPERSHAPE was not a functional tool for them. Although they were prepared to *use* the procedure when they were asked to do so explicitly by the researchers, they reverted to direct drive at the earliest opportunity. Finally, we noted in passing that the two boys in the study were completely prepared to ignore the finer points of the visual outcome of their procedures -- a finding which contrasted strongly with that of the girls.

References

Hoyles C. & Noss R. (1988) Children working in a structured Logo environment: from doing to understanding. Récherches en Didactiques de Mathématiques, **8**.

Hoyles C. & Noss R. (1987) Seeing what matters: developing an understanding of the concept parallelogram through a Logo microworld. Procs. of PMEXI, Montreal.

One mathematics teacher

Barbara Jaworski - Open University - U.K.

Nature and purpose of the study

This in-depth study of one teacher is part of a wider study of a number of teachers which aims to explore their mathematics teaching in all of its facets, including:

- their beliefs, about mathematics, cognition, teaching and learning;
- their ways of interacting with pupils in the classroom;
- their devising and presentation of activities for pupils;
- their classroom organisation and management;
- their assessment and evaluation of pupils mathematical learning;
- their assessment and evaluation of their own work.

Its purpose is to find out more about what mathematics teaching implies and involves, and perhaps about how teaching can be more closely related to the learning of the pupils. I present only one teacher, Clare, in this report because it would be impossible to do justice to more than one in the space and time available, and because an understanding of the study as a whole depends upon an appreciation of the nature and depth of the data collected.

Methodology and data

Clare was involved in the second phase of the project. The methodology here is substantially that of Case study form of a substantially ethnographic nature with participant observation and some interviewing as discussed in Stenhouse [1]. The first phase had been one of exploring what might be involved in in-depth research into teachers' classroom practice in mathematics and of evolving a methodology. Two teachers were involved. The developing methodology was then employed with another two teachers in the second phase. This involved me, the researcher, in:

1) Discussion with the teacher about her lesson intentions.
2) Participant-observation of a lesson and recording by hand-written field notes.
3) Audio recording of aspects of certain lessons.
4) Video recording of aspects of certain lessons.
5) Discussion with the teacher after a lesson about what had occurred, her own perceptions of it and her comments on the researcher's perceptions of it.
6) Obtaining written comments from the teacher about audio or video material from her classroom, and talking with her about aspects of this material.
7) Discussions with the teacher about mathematics teaching generally, about issues with which she was concerned and about her own students and their learning.

8) Conversations with some of the teacher's students.
9) Eliciting students' attitudes and opinions through interviews and questionnaires.
10) Talking with the teacher and her colleagues about their teaching, sometimes with video recordings of certain of their classrooms as a stimulus.

Qualitative data was obtained in various forms: field notes; audio and video recordings from the classroom and transcripts of these; audio recordings and transcripts of conversations between teacher and researcher; audio recordings and transcripts of pupil interviews; questionnaires from pupils; video recordings and transcripts of teacher group discussion.

Circumstances particular to research with Clare

Clare, who had been teaching for about seven years, was a competent teacher who was recognised conventionally as being successful. She taught mathematics in a comprehensive school of 12-18 year old pupils. Most observation and discussion concerned one mixed ability class of 24 fourth-year pupils (aged 15) who remained in this class for all of their lessons. Another of Clare's classes was also observed and discussed and all of her classes completed a prepared questionaire. Classroom observations occurred once or twice per week over two and a half terms. Discussions were fitted in before and after lessons and at specially arranged times outside school hours. As a result of all of this I built up a mental picture of Clare as a mathematics teacher which I have tried to express and defend with reference to the data which I collected.

Beliefs behind and implications of this methodology

It is not possible to know objectively either what occurs in a lesson or the reasons for it as all observation involves interpretation. To speak rationally about what occurred and why, the researcher needs not only to observe the event but to get as close as possible to understanding the teacher's perception of the event. This involves a dilemma:

In understanding the teacher's perception, the researcher needs to act as distancer, helping the teacher to separate her reflective self from her active self (Schon [2]) in an effort to analyse better her actions and thinking in the classroom. This analysis requires self-awareness, self-honesty and analytical persistence on the part of the teacher, and the researcher can encourage these by asking appropriate questions, urging further consideration and offering support and encouragement. The act of distancing is best possible when the agent is separate from involvement in the action and thinking; thus the teacher, being intimately involved, finds it hard to be the distancing agent for herself. The researcher begins the act in this separated position, but the very nature of her intention in undertaking the act, which is to get closer to the thinking of the teacher, draws her into the web and reduces her distancing capabilities.

Thus the researcher has to be careful with interpretations which are based on the teacher's perceived perceptions. From a distance she may be misinterpreting the teacher's words, but as she becomes closer in understanding to the teacher she may lose the ability to encourage the teacher to question her own interpretations. The analysis which follows must be viewed in the light of these remarks.

Analysis

It needs to be said that the form of this analysis was not obvious and the doing of it was not easy. I wanted firstly to characterise Clare as a mathematics teacher, and secondly to produce a characterisation that was in some sense generalisable. I wrote down many attributes and many descriptive categories. I tried to substantiate my descriptions with events and quotations. I found my categorisations indistinct and elusive.

For example when working at the board on some aspect of fractions, Clare said to the class, "Anyone who's ahead of this, try to think how to explain the repetition in 1/7". In one respect this is classroom management. Discussion was on points of difficulty which some students were experiencing while others seemed to understand and were possibly getting bored. This comment enabled them to make progress while Clare gave her attention to the others. In another respect it shows the level of challenge in her instructions to students - "try to think how to explain" was typical Clare-speak, and it was to her credit that students seemed not to be worried by such complex instructions.

One brief comment being so rich in interpretation illustrates the complexity of the task. I decided that I was trying to distinguish too finely and that what I needed in the first instance was a much broader brush so I settled on broader categories which seemed to encapsulate Clare's qualities. Due to limitations of space I have chosen to concentrate on just three of these which have emerged strongly from my observations of and discussions with her. I was influenced by the reporting and analysis of data in Fensham et al [3].

1. Classroom management and management of learning

Clare is strongly, even forcefully, in charge of what happens in her classroom. Her expectations are both explicit and implicit in what occurs. Students respond favourably to this, recognising its value while ruefully admitting that they might choose it to be otherwise. She is most concerned with encouraging students to think about what they are doing

and why they are doing it and to organise their own work and thinking.
Some quotations from her instructions in the classroom:

"Today we'll work on KMP (their maths scheme). We'll have two lessons on this, so plan your work."

"How many people have calculators? It's a good idea to bring them to all lessons"

"Think! - no, I mean a hands-down think."

"In order to get this off the ground, can we have just one person speaking at a time. Because if you think that what you have to say is valuable then it is probably going to be valuable to everyone."

A boy complains, rather agressively, that he doesn't know what to do. Referring to the task set, he says "I've done this before." Clare replies, "I don't ask you to waste your time - don't treat it like that".

At the end of a class project on 'pentominoes' she told the class that they should hand in their written report after the next maths lesson. "So,", she said, "this is the lesson to see me and ask me about it". She went on, "But if you want a solution, I'm not going to give you one. There's nothing wrong with handing in a project where you haven't found an answer. If I tell you, then you won't get that kick from having found it yourself"

In many of the lessons which I observed, students chose where to sit and with whom, but occasionally Clare directed them into specific positions or groups. "Jerome, come and sit here please. I want you to work on your own today, 'cause I want to find out what you think about ....". Daisy, will you work with John and Stephen today please, because I think you're all thinking along the same lines ...". She disagreed strongly with one of her colleagues who claimed that friendship groups were the best form of organisation as they provided a secure and supporting environment in which students could work. Clare believed that students needed to work in different situations and with different people for stimulation and to gain a variety of experiences rather than always relying on a protective situation. I observed that relationships within the class were mostly good and that students did not in general seem to mind with whom they worked.

In one project where students were gathering information about population distributions she said, "All groups .. pool what you've found and decide what questions you want to ask next". And on another occasion, again to the whole class, "In about 3 minutes I want some feedback from you. Just think about what you're going to say."

2. Sensitivity to students and their individual needs

Although her decisiveness and personality occasionally verged on the formidable, Clare was also caring and sensitive to students' individual needs and characteristics. She was never unapproachable, and students tended to treat her with a familiar respect. She maintained an informal and often jocular relationship with them.

She wrote for me on one occasion, 'The students and I know each other well. There is trust and humour on all sides and they understand that in the joyful melee of mixed-ability teaching, I will sometimes be lost for words, in a muddle, badly tuned, or just plain wrong.'

Much of our recorded dialogue consists of her comments on particular students: Daisy and Naomi who are bright but stuck in a rut and need to be stimulated; Jacques, who is bright but in trying to cut corners does not do full justice to his thinking; John who has 'maverick ideas' but has difficulty in following them up; Annette who is totally lacking in confidence and needs to experience some success; Frances who has such overwhelming difficulties that Clare despairs of ever being able to help her; Jerome who is lazy and will rarely make any effort. I have pages and pages of notes on these and others, and feel that I know them well myself through Clare's descriptions. After a particular interview which I had with two students, Clare reported one of them as asking in a wondering tone, "How does she know so much about us?"

It is typical of Clare to get excited about or to agonise over particular students at length. For example she said on one occasion, "I have a student in the foundation year who has a slightly embarrassing stutter and really can't read, or write, very well. She is one of the brightest, most creative, mathematicians in that group. When I said brightest, that's probably not what you could measure in a test, it's not like that sort of bright, but she's one that I can rely on to make the classroom come alive, and work ...yes, in an illuminating sense. And she comes up with ideas - the sort of person who will invent things. I mean, she invented this morning the prime factor rectangle and the factor prime rectangle. She said, 'Is it alright for me to invent a prime factor rectangle?' and I said, 'If you can tell me what it is, yes.' You know she's just so sort of open and creative about the subject.

In our discussion after one lesson, I had asked Clare if she had noticed a girl, Virginia, sitting with her hand up for quite some time. She replied, "Yes, she did. It was quite a good lesson for Virginia because she doesn't always take any part at all, and she was actually working very well this morning." When Clare later listened to the audio recording of the lesson and the discussion, she wrote as a comment: 'I sidestepped Barbara's comment about V's hand up by saying she was working well. If she was waiting for me with her hand up she wasn't working, and it was my fault! Guilt!! I hadn't really been on the lookout for hands up during the lesson. I hope I haven't let Virginia down.'

3. Challenging the students mathematically

Clare expressed on many occasions her struggle with helping students to develop their own mathematical ideas and concepts rather than just accepting mathematics from her.

"Naomi .. she's very bright.. but she couldn't divide 6 by 4/5. I wasn't going to tell her! But I couldn't think of how to tell her how to divide fractions."

When Frances and Joanne had come up with different results, Clare said to each of them in turn, "You try to convince Frances that you're right. You try to convince Joanne that you're right.

In response to a conjecture made by a pair of girls she asked, "What are you going to do to find out if that's true?"

Many of Clare's lessons involved project work in which students were asked to investigate some given situtation. In one example they explored the under and over patterns in a piece of string when it was dropped onto the table, and whether, by pulling both ends it was possible to form a knot. Clare was very aware of her propensity to 'push and prod' and felt that when she had particular ideas or results in her head, she was likely consciously or unconsciously to push students towards them. She said after one lesson on Knots, "The way I work with these things is that if I know too much about where it's going, given that I do prod and guide, I may well prod and guide people into directions which may not be most fruitful ones, may not be the most interesting ones to them." In trying to expand on this and justify her thinking and motivation she later wrote, "It sounds as if 'anything goes', but I only feel 'anything within certain limits goes'. I will know the limits when I reach them."

We watched a piece of video together of Clare working with one student, Annette, on a workcard on area. The sound was particularly poor and Clare stopped the tape at one point to explain, "This is a lovely conversation - it's all about chopping up. It's an L shape, chopped into two rectangles, and she actually realises why she has to chop it into two rectangles. At one point she told me why to chop it into two rectangles to get the figure, and then when I ask her to do it, she does it totally differently. Instead of having the L-shape chopped into two rectangles, she actually makes it into a bigger rectangle. So I think, Oh Hell, but we'll take her through this, and she get's there! She can't tell me what she is going to do, but then she does it exactly right - it's ever so exciting!"

Triangulation - students comments

There has often been a considerable period of time between the collection of the data and analysis of it. Very often in the analysis questions arise where it would have been nice to obtain students' comments. For example, regarding the lesson where Virginia had her hand up for a period of time without Clare noticing, it may have been helpful to have asked Virginia about her feelings and reactions to being ignored. However, apart from occasional conversations with students which happened spontaneously, all data from students came from arranged interviews and questionaires at the end of my period of work with Clare.

Some of the interviews produced unsolicited comments about Clare's way of working which very strongly supported what I was seeing and what Clare claimed to be her purpose. For example in response to the question: "What do you think about the way Clare runs lessons? About the organisation and the things she expects you to do or not do?", one boy replied:

"Well she's basically very strict. It's a funny sort of strictness because it's not sit down and quietness and this, because she allows a certain amount of leeway. So I mean she will let you sit with your friends when you start off, and chat, but sooner or later she decides, you know, if it's good for you. I think that Clare wants you to get the best of your capabilities, that she is continually pushing you, in some ways - in most ways it's good, but I have found once or twice that it tends to worry you, you know you haven't done enough, or you are not doing enough, and you have all the other subjects to worry about."

Another student said, "She seems to be pushing you along, you know, because I think she sees your capabilities more than you do."

At another time, in reply to a question about similarities and differences between maths and other subjects a student said, "I think in Maths, especially with Clare people do more work in the class as a whole. She is a much stricter teacher and she really pushes you forward, to get your goal, to the height of your ability. So I think a lot of people are doing quite well in maths because she is always there to give you that extra push and makes you go further.

Again on the subject of how maths is different, one student said, "I think maths is different because everybody sort of works ... with people talking I find it much harder to work." I then asked, "Is that something to do with the way Clare runs the lessons, or is it just because it's maths?", and the reply was, "Yes, I think it is because of Clare, you know, if they talk they get into trouble, or get moved.

Conclusions

Care needs to be taken in generalising from interpretations of qualitative data of this sort (Stenhouse [1] : Cohen & Manion [4]). However, when I have applied the same level of analysis to the second teacher of this phase of my study I hope to be able to make links between the findings on the two teachers and possibly link back to the two teachers in the first phase. I should like to explore whether the differing emphases of the teachers correspond to differences in their classrooms, whether common beliefs or strategies correspond to similar effects, whether there is any agreement that particular ways of working promote 'successful mathematical learning' and how that is seen to be defined, whether the ways the teachers see themselves developing have any common features. I hope to form some conjectures and questions which I can take into the third phase for testing. Ultimately I should like to be able to make some general statements related to the facets listed on page one.

I expect the third phase of the study to be different to the first two in the following respects:

1) I wish to enter the third phase with well defined questions which I want to pursue.
2) I hope to modify my methodology to improve on deficiencies in the second phase. For example I hope to interview students closer to the event to allow more student input at the fine level of data collection.
3) I wish to explore how my own beliefs affect the teacher's responses and actions. Both of the teachers in the second phase have indicated words or opinions of mine which have influenced their thinking and I should like to pursue this more overtly.
4) Related to (3). The relationship between teacher and researcher has been fruitful according to teachers in the second phase. I should like to look deeper into the implications of this relationship.

References

[1] Stenhouse L. Case study & Case study records. British Educational Research Journal 4(2) 1978
[2] Schon D. The Reflective Practitioner, Temple Smith, London 1983
[3] Fensham P., Power C., Tripp D., Kemmis S. Alienation from Schooling Routledge & Kegan Paul, London 1986
[4] Cohen L. & Manion L. Perspectives on Classrooms and Schools, Holt Rinehart & Winston, 1981

LEARNING THE STRUCTURE OF ALGEBRAIC EXPRESSIONS AND EQUATIONS

Carolyn Kieran
Université du Québec à Montréal

Abstract. This theoretical paper begins with a brief discussion of the meaning of "structure", within the context of the early part of the high school algebra course. Students' difficulties with learning the structural aspects of algebra are shown by examples from several cognitively-oriented, research studies. The paper concludes with some suggestions for algebra instruction.

The teaching of high school algebra usually begins with the topics: variables, simplification of algebraic expressions, equations in one unknown, and equation solving. Students' difficulties with these topics have been found to center on (a) the meaning of letters, (b) the shift to a set of conventions different from those used in arithmetic, and (c) the recognition and use of structure. Since the first two of these difficulties have already been well documented in the research literature (e.g., Booth, 1981, 1984; Kuchemann, 1981; Matz, 1979), this paper reviews some of the research literature related to the third one--recognition and use of structure. Because of space constraints, it is not possible to review in this paper all of the pertinent literature; a more complete description can be found in Kieran (in press b).

Structure

The term "structure" is used in many different contexts throughout this paper. In a general sense, we refer to "arithmetic/algebraic structure" as a system comprising a set of numbers/numerical variables, some operation(s), and the properties of the operation(s). However, we also refer in this paper to particular aspects of structure, such as the structure of expressions and the structure of equations.

"Structure" is defined by Webster to mean "the arrangement of the parts in a whole, the aggregate of elements of an entity in their relationships to each other." The former deals with arrangement or disposition; the latter with relationships. When we speak of the structure of an algebraic or arithmetic expression, we mean both (a) the *surface* structure, which refers to the given form or disposition of the terms and operations, subject when disposed sequentially to the constraints of the order of operations; and also (b) the *systemic* structure (systemic in the sense of relating to the mathematical system from which it inherits properties), which refers to the properties of the operations such as commutativity and associativity, and the relationships between the operations such as distributivity. The systemic structure of algebraic expressions permits us to express, for example, $3(x + 2) + 5$

equivalently as $5 + 3(x + 2)$ or as $3x + 11$ and so on. Thus, the structure of the expression $3(x + 2) + 5$ comprises the surface structure, that is, the given ensemble of terms and operations--in this case, the multiplication of 3 by $x + 2$, followed by the addition of 5-along with the systemic structure, that is, the equivalent forms of the expression according to the properties of the given operations.

The structure of an equation incorporates the characteristics of the structure of expressions, for an equation relates two expressions. Thus, the surface structure of an equation comprises the given terms and operations of the left- and right-hand expressions, as well as the equal sign denoting the equality of the two expressions. Similarly, the systemic structure of an equation includes the equivalent forms of the two given expressions. For example, the equation $3(x + 2) + 5 = 4x/2 - 7$ can be re-expressed as $3x + 11 = 2x - 7$, wherein each expression is independently transformed (i.e., simplified). Because of the equality relationship inherent in an equation, the left-hand expression continues to be equivalent to the right-hand expression after such systemic transformations of one or both expressions. The resulting equation is also equivalent to the given equation. However, the systemic structure of an equation comprises much more than the systemic structure of expressions. Because of the equality relationship and system properties such as the addition property of equality ("if equals are added to equals, the sums are equal"), the equation as a whole can be transformed into equivalent equations without necessarily replacing one or both expressions by equivalent ones. For example, the equation $3x + 11 = 2x - 7$ is equivalent to the equation $3x + 11 + 7 = 2x - 7 + 7$, even though the left-hand expression $3x + 11$ is not equivalent to $3x + 11 + 7$, nor is the right-hand expression $2x - 7$ equivalent to $2x - 7 + 7$. Similarly, the equation $5x + 6 = 10$ is equivalent to $5x = 10 - 6$, according to the properties of our arithmetic/algebraic system, wherein an addition can be expressed as a subtraction. The system properties of equality can be used to generate an infinite set of equations, in fact, a class of equivalent equations. It is this particular aspect of the systemic structure of equations--that is, the potential of generating equivalent equations by means of properties related to (a) performing the same operation on both sides of an equation, and (b) the alternate ways of expressing additions and multiplications in terms of subtractions and divisions--that is so crucial to the process of solving equations.

Variables

High school algebra usually starts with instruction in the concept of variable--a prerequisite to understanding the systemic structure of algebraic expressions and equations. In elementary school, children have already seen placeholders in "open sentences" (sometimes called missing addend problems), and have used letters in formulas such as the area of a rectangle. However, their past experiences cannot easily be related to the many uses of variable to which they are exposed in high school algebra. In a large-scale study of some of the various ways in which high school students use algebraic letters, carried out by Kuchemann (1978, 1981), it was found that most students could not cope consistently with questions that required the use of a letter as a specific unknown. The findings of a follow-up study, the Strategies and Errors in Secondary Mathematics (SESM) project, reported by Booth (1984), suggest that some of the difficulty which students have in interpreting letters as representing generalized numbers may be related to a 'cognitive readiness' factor: The lower ability mathematical groups were unable to evolve in their interpretation of letters as did the middle and top ability groups. Another finding of the SESM study was that, even though beginning algebra students are initially unreceptive to the idea of unclosed, non-numerical answers (such as $x + 3$), instruction can be quite effective in changing their thinking in this regard.

Algebraic Expressions

After being introduced to the notion of using letters to represent numbers, the next topic in the algebra programme is usually operating with these letters in the context of simplifying algebraic expressions (e.g., $2x + 3y$). Chalouh and Herscovics (in press) carried out a teaching experiment (six children, 12 to 13 years of age) in which they investigated the cognitive obstacles involved in constructing meaning for algebraic expressions when using a geometric approach. In designing their teaching experiment, they took into consideration the work of Collis (1974) and of Davis (1975) concerning the incongruencies between arithmetic and algebra, the consequent inability of novice algebra students to regard algebraic expressions as legitimate answers, and the difficulties they experience with algebraic concatenation. Chalouh and Herscovics used an instructional sequence that included arrays of dots, line segments, and areas of rectangles. The lessons permitted the children to develop meaning for expressions such as $2a + 5a$, but most of the children were not able to interpret this expression as $7a$. This study showed that constructing meaning for algebraic expressions does not necessarily lead to spontaneous development of meaning for the simplification of algebraic expressions.

While the above study emphasized children's construction of meaning for the form of algebraic expressions, other studies (e.g., Greeno, 1982) have investigated children's structural knowledge of these expressions as evidenced by the processes they use to simplify them. Greeno (1980) has suggested that the process of solving problems involves apprehending the structure of relations in the problem. To test this idea, he carried out a study with beginning algebra students on tasks involving algebraic expressions (Greeno, 1982). He found that their performance appeared to be quite haphazard, for a while at least. Their procedures seemed to be fraught with unsystematic errors, thus indicating an absence of knowledge of the structural features of algebra. Their confusion was evident in the way that they partitioned algebraic expressions into component parts. According to Greeno, beginning algebra students are not consistent in their approach to testing conditions before performing some operation, nor with the process of performing the operations. For example, they might simplify $4(6x - 3y) + 5x$ as $4(6x - 3y + 5x)$ on one occasion, but do something else on another occasion.

Algebraic Equations and Equation Solving

Students' difficulties with apprehending the structure of algebraic expressions carry over into their work with the next topic of the programme, algebraic equations (e.g., $2x + 3 = 7$). One of the findings of the Algebra Learning Project (Wagner, Rachlin, & Jensen, 1984) was that algebra students have trouble dealing with multiterm expressions as a single unit: Students appeared not to perceive that the basic surface structure of, for example, $4(2x + 1) + 7 = 35$, was the same as for $4x + 7 = 35$.

A recent study with a teaching component (Thompson & Thompson, 1987) has shown that instruction can improve students' ability to recognize the form or surface structure of an algebraic equation. These researchers designed a teaching experiment involving two instructional formats: algebraic equation notation and expression trees displayed on a computer screen. After instruction, their eight 7th-grade subjects (12-year-olds) did not overgeneralize rules, nor did they fail to adhere to the structure of expressions. They also developed a general notion of variable as a placeholder within a structure and the view that the variable could be replaced by anything: a number, another letter, or an expression.

A teaching experiment carried out by Herscovics and Kieran (1980) emphasized another aspect of the structure of an algebraic equation: the equivalence of left- and right-hand expressions. In a series of individual sessions, six 7th-grade and 8th-grade children were guided in constructing meaning for equations in which each expression did not contain simply a numerical term (i.e., for equations with the surface structure $ax \pm b = cx \pm d$). The instructional sequence began with an extension of the notion of arithmetic equality to include equalities with more than one numerical term on the right side and then went on to hiding the numbers of these "arithmetic identities." This approach was found to be accessible to these algebra novices and effective in expanding their view of the equal sign from a "do something signal" (Behr, Erlwanger, & Nichols, 1976) to that of a symbol relating the value of the left-hand expression with that of the right-hand expression (Kieran, 1981).

Many studies have focused on students' knowledge of parsing (i.e., recognition of the surface structure of an expression or equation). Davis (1975), Davis, Jockusch, and McKnight (1978), Matz (1979), Greeno (1982), and others have all shown that beginning algebra students have enormous difficulties in imposing structure on expressions involving various combinations of operations, numerical terms, and literal terms. Parsing errors, such as simplifying $39x - 4$ to $35x$, have been documented in several studies. These same errors have been found to persist among college students (e.g., Carry, Lewis, & Bernard, 1980).

Another facet of arithmetic/algebraic structure concerns the relationship between the operations of addition and subtraction (and between multiplication and division) and the equivalent expressions of these relationships (e.g., $3 + 4 = 7$ and its equivalent expression $3 = 7 - 4$). Knowing these relationships and their written forms could conceivably enable a student to see that $x + 4 = 7$ and $x = 7 - 4$ are equivalent and that they have the same solution. However, such may not be the case: A group of six twelve-year-old beginning algebra students showed considerable confusion over equations involving the addition-subtraction relationship (Kieran, 1984). This was seen with two of their errors: the Redistribution error and the Switching Addends error. In the Switching Addends error, $x + a = b$ was considered to have the same solution as $x = a + b$; in the Redistribution error, $x + a = b$ was considered to have the same solution as $x + a - c = b + c$. In this last equation, the subtraction of c on the left was balanced by the addition of c on the right.

Another aspect of structural knowledge considered to be important in equation solving involves knowledge of equivalence constraints. Greeno (1982) has pointed out that algebra novices lack knowledge of the constraints which determine whether transformations are permissible. For example, they do not know how to show that an incorrect solution is wrong, except to re-solve the given equation. They do not seem to be aware that an incorrect solution, when substituted into an incorrectly transformed equation will yield different values for the left and right sides of the equation. Nor do they realize that it is only the correct solution which will yield equivalent values for the two expressions in any equation of the equation-solving chain.

Students' understanding of equation structure, as related to the solution of an equation, was also investigated in the Kieran study (1984). The six novices were presented with pairs of equations and were asked whether or not the equations had the same solution, without actually solving the equations. The method the students used was to compare the two equations, attempting to pick out what did not match and, on the basis of their arithmetical knowledge, to determine whether the mismatches were legal or not. In scanning the equation-pairs for similarities and differences, the novices followed a left-to-right search pattern and rarely seemed to be able to take in all of the differences between the equations. This inability of beginning algebra students to discriminate the essential features of equations has important consequences for learning theory.

Another large body of research exists in which the focus has been on the procedures used by novices in the solving of equations. Some of these studies have included different "concrete" modeling techniques as a method of helping students construct meaning for certain forms of equations and for the operations carried out on these equations. One such study was carried out by O'Brien (1980) who worked with two groups of twenty-three 3rd-year high school students. One group was taught meaning for equations and for the manipulations performed on equations by means of concrete materials (bundles of counters and colored cubes). The manipulations involved removing objects from both sides or adding objects to both sides of the concretely-modeled equation. The second group was taught meaning for manipulations using a generalization of the part-whole strategy (i.e., the relationship between addition and subtraction--$2 + 3 = 5$ compared with $2 = 5 - 3$), often called the "Change Side"-"Change Sign" rule. O'Brien found that the second group became more proficient equation solvers than the concrete materials group.

Concrete models have also been used in teaching experiments by Filloy and Rojano (1985a, 1985b) in studies aimed at helping students create meaning for equations of the type $A\underline{x} \pm B = C\underline{x}$ and for the algebraic operations used in solving these equations. Their main approach was a geometric one, although they also used the balance model in some of their studies. Teaching interviews with three classes of 12- and 13-year-olds who already knew how to solve equations of the types $\underline{x} \pm A = B$ and $A\underline{x} \pm B = C$ showed that the use of these two concrete models (the balance and the area models) did not significantly increase most students' ability to operate at the symbolic level with equations having two occurrences of the unknown. The well known equation-solving error of combining constants and coefficients was also seen in this study, in particular with the use of the geometric model. Students tended to fixate on the model and seemed unable to apply previous equation-solving knowledge to the simplified equations of the instructional sequence.

A final study to be discussed in this section on equation solving is one which did not use concrete models but rather drew on the numerical approach used in an earlier teaching experiment by Herscovics and Kieran (1980). At the outset of the study, Kieran (in press a) pretested six average-ability 12-year-olds who had not had any previous algebra instruction. She found that the students showed two different equation-solving preferences, both based on their elementary school experience with 'open sentences.' Some preferred to solve the simple equations of the pretest by means of arithmetic methods such as substitution and known number facts; others preferred inversing, that is, solving $2x + 5 = 13$ by subtracting 5 and then dividing by 2 (and in fact seemed unaware of the potential of substitution as an equation-solving procedure). Those who preferred substitution viewed the letter in an equation as representing a number in a balanced equality relationship; those who preferred inversing viewed the letter as having no meaning until its value was found by means of certain transposing operations. (See Kieran, 1983, for more details on these students' views of algebraic letters.) In the teaching experiment on equation solving which followed, the procedure of performing the same operation on both sides of an algebraic equation was carried out first on arithmetic equalities (e.g., $10 + 7 = 17$), and then on the algebraic equations built from these arithmetic equalities (e.g., $x + 7 = 17$):

$$\begin{array}{c|c} 10 + 7 = 17 & x + 7 = 17 \\ \downarrow & \downarrow \\ 10 + 7 - 7 = 17 - 7 & x + 7 - 7 = 17 - 7. \end{array}$$

Kieran found that those students who had initially preferred inversing (i.e., transposing) were in general unable to make sense of the solving procedure being taught, that is, performing the same operation on both sides of an algebraic equation. This suggests that, although inversing is considered by many mathematics educators to be a shortened version of the procedure of performing the same operation on both sides, these two procedures may be perceived quite differently by beginning algebra students. The procedure of performing the same operation on both sides of an equation emphasizes the symmetry of equations; this emphasis is quite absent in the use of the procedure of inversing. Although this investigation involved only six case studies of beginning algebra learners, the findings suggest that there may not be just one path which is followed in the learning of algebra. Some learners focus initially on the given surface operations and on the relationship of equality between left- and right-hand expressions of an equation; they may be more open to the solving procedure of performing the same operation on both sides. Other learners focus immediately on transposing and on the inverses of the given surface operations; they may prefer to solve equations, not by the same-operation-to-both-sides method, but by extending their transposing method.

III Concluding Remarks

The early learning of algebra involves grappling with the topics of variables, algebraic expressions, equations, and equation solving. The research discussed in this paper has shown that students have difficulty with recognizing and using the structure of introductory algebra. It has been found that some aspects of this difficulty are amenable to instruction; others less so. One particularly troublesome area concerns the understanding of a particular feature of algebraic structure--the equality relationship between left- and right-hand expressions of equations. This relationship is a cornerstone of much of the algebra instruction currently taking place. It is the basis of many of the concrete models used to represent equations and equation solving; it is also an integral part of the symmetric procedure of performing the same operation on both sides of the equation. However, it has been found that for some students, teaching methods based on this aspect of the structure of equations often do not succeed. For these students, who tend to view the right side of an equation as the answer and who prefer to solve equations by transposing, the equation is simply not seen as a balance between right and left sides, nor as a structure that is operated on symmetrically. That understanding seems clearly to be absent. These same students also appear to have difficulty in formalizing even such simple relationships as the equivalent forms of addition and subtraction. Another finding of many of the studies discussed in this paper concerns the inability of beginning algebra students to "see" the surface structure of algebraic expressions which contain various combinations of operations and literal terms. This difficulty seems to continue throughout the algebra career of many students, as evidenced by errors such as reducing $(a + b + c)/(a + b)$ to c, seen among college students. In conclusion, many high school students appear to be experiencing serious obstacles in their ability to recognize and use the structure of school algebra. The challenge to researchers is to devise studies that will push forward our knowledge of how students can come to understand the structure of elementary algebra and of algebraic methods.

References

Behr, M., Erlwanger, S., & Nichols, E. (1976). *How children view equality sentences* (PMDC Technical Report No. 3). Florida State University. (ERIC Document Reproduction Service No. ED144802)

Booth, L. R. (1981). Child-methods in secondary mathematics. *Educational Studies in Mathematics*, *12*, 29-41.

Booth, L. R. (1984). *Algebra: Children's strategies and errors* Windsor, Berkshire: NFER-Nelson.

Carry, L. R., Lewis, C., & Bernard, J. (1980). *Psychology of equation solving: An Information Processing study* (Final Technical Report). Austin, TX: University of Texas at Austin, Department of Curriculum and Instruction.

Chalouh, L., & Herscovics, N. (in press). Teaching algebraic expressions in a meaningful way. In A. Coxford (Ed.), *1988 Yearbook of National Council of Teachers of Mathematics: Algebraic concepts in the curriculum, K - 12*. Reston, VA: NCTM.

Collis, K. F. (1974, June). *Cognitive development and mathematics learning*. Paper presented at the Psychology of Mathematics Workshop, Centre for Science Education, Chelsea College, London.

Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. *Journal of Children's Mathematical Behavior*, *1*(3), 7-35.

Davis, R. B., Jockusch, E., & McKnight, C. (1978). Cognitive processes in learning algebra. *Journal of Children's Mathematical Behavior*, 2(1), 10-320.

Filloy, E., & Rojano, T. (1985a). Obstructions to the acquisition of elemental algebraic concepts and teaching strategies. In L. Streefland (Ed.), *Proceedings of Ninth International Conference of PME*. Noordwijkerhout, The Netherlands.

Filloy, E., & Rojano, T. (1985b). Operating on the unknown and models of teaching. In S. Damarin & M. Shelton (Eds.), *Proceedings of Seventh Annual Meeting of PME-NA*. Columbus, Ohio.

Greeno, J. G. (1980). Trends in the theory of knowledge for problem solving. In D. T. Tuma & F. Reif (Eds.), *Problem solving and education: Issues in teaching and research* (pp. 9-23). Hillsdale, NJ: Erlbaum.

Greeno, J. G. (1982, March). *A cognitive learning analysis of algebra*. Paper presented at annual meeting of American Educational Research Association, Boston.

Herscovics, N., & Kieran, C. (1980). Constructing meaning for the concept of equation. *Mathematics Teacher*, *73*, 572-580.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*, 317-326.

Kieran, C. (1983). Relationships between novices' views of algebraic letters and their use of symmetric and asymmetric equation-solving procedures. In J. C. Bergeron & N. Herscovics (Eds.), *Proceedings of Fifth Annual Meeting of PME-NA*. Montreal, Quebec.

Kieran, C. (1984). A comparison between novice and more-expert algebra students on tasks dealing with the equivalence of equations. In J. M. Moser (Ed.), *Proceedings of Sixth Annual Meeting of PME-NA*. Madison, Wisconsin.

Kieran, C. (in press a). Two different approaches among algebra learners. In A. Coxford (Ed.), *1988 Yearbook of National Council of Teachers of Mathematics: Algebraic concepts in the curriculum, K - 12*. Reston, VA: NCTM.

Kieran, C. (in press b). The early learning of algebra. In C. Kieran & S. Wagner (Eds.), *Research issues in the learning and teaching of algebra*. Reston, VA: National Council of Teachers of Mathematics.

Kuchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in School*, *7*(4), 23-26.

Kuchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16*. London: John Murray.

Matz, M. (1979). *Towards a process model for high school algebra errors* (Working Paper 181). Cambridge, MA: Massachusetts Institute of Technology, Artificial Intelligence Laboratory.

O'Brien, D. J. (1980). *Solving equations*. Unpublished master's thesis, University of Nottingham, England.

Thompson, P., & Thompson, A. (1987). Computer presentations of structure in algebra. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of Eleventh International Conference of the Psychology of Mathematics Education* (pp. 248-254, Vol. 1). Montreal, Quebec, Canada: Université de Montréal.

Wagner, S., Rachlin, S. L., & Jensen, R. J. (1984). *Algebra Learning Project: Final report*. Athens, GA: The University of Georgia, Department of Mathematics Education.

THE INFLUENCE OF TEACHING ON CHILDREN'S STRATEGIES FOR SOLVING PROPORTIONAL AND INVERSELY PROPORTIONAL WORD PROBLEMS

Wilfried Kurth, Universität Osnabrück, W.-Germany

Prior to the teaching unit, children are left to strategies of their own, when they try to solve proportional and inversely proportional word problems. With the help of a test, several succesful and error strategies were found. During the teaching unit, children have learnt to relate word problems with the concepts "proportion" and "inversely proportion" and to solve them by using characteristic peculiarities of these types of function.
In this way, the children become more succesful in general, but the different types of error decrease to a different extent, some don't decrease. One type of error - to take an inversely proportional problem for a proportional one - even increases distinctly. The results of the investigation are presented and then are tried to be interpreted.

The solution of proportional and inversely proportional word problems is mainly taught in the seventh grade in schools of all types in the F.R.G. (age of the pupils about 12 years). The aims of the corresponding teaching unit are

- the ability to gather from the text whether the function is proportional or inversely proportional (or neither)
- the ability to solve the problem by applying a procedure that corresponds to the respective function. This procedure (e. g. the rule of three, the method of fraction operators, fractional equations) is usually introduced as a schematic procedure, i. e. the rules applied are presented in a particular optical fashion.

A typical kind of problem is the missing value-problem which requires the calculation of a forth value on the basis of three given ones.

The children already know the arithmetical operations (multiplication and division of rational numbers) for solving proportional and inversely proportional problems. The question is, how far this knowledge will help pupils to succeed in working out a strategy referring to the situation presented in the text of the excercise.

Within our investigations we are mainly engaged in finding out which strategies and types of error are produced by the pupils before and after the teaching unit and how these changes can be explained.

The investigations are composed of a preliminary test (before the teaching unit), teaching observations in some classes, and a follow-up test identical to the first one (about 6 weeks after the teaching unit). Additionally, we interviewed some pupils in order to get more information on their problem-solving-process. 217 pupils from 11 classes of the "Realschule" (the secondary school within the tripartite school system of the FRG) were involved in the investigation outlined here.

The test consists of 10 missing-value-problems (5 proportional and 5 inversely proportional). Previous investigations (Hart 1981, Karplus et al. 1983, Kurth 1987, Noelting 1980) showed that pupils adjust their strategies very much to the chosen ratios, i. e. possible calculation difficulties influenced the extraction of the operations from the text of the exercise. If the three given values a, b, c and the unknown "x" are arranged in a table (M_1, M_2 are the two measure spaces)

M_1	M_2
a	b
c	x

the five following combinations of ratios are included (a, b, c, x integer):

1) c:a and b:a both integer
2) c:a integer, b:a not integer
3) c:a not integer, b:a integer

4) $c:a = 3:2$, $b:a \neq 3:2$, $b<a$

5) $c:a$, $b:a$ both not integer, both not equal 3:2, $c<a$, $b<a$.

Each of the five combinations refers to one proportional problem (no. of evaluation: 1-5) and one inversely proportional problem (no. of evaluation: 6-10) occuring in the test in a mixed order:

Test exercises according to the order of evaluation:

1. Out of 7 liters of milk, you can make 42 grams of butter. How many grams of butter would you get from 21 liters?
2. Within 5 days, a potato-chip factory uses 6 truck loads of potatoes. How many truck loads of potatoes would the factory use within 30 days?
3. In 12 seconds, a water pump can fill 36 liters of water into a pool. How many liters of water can the same pump fill into the pool in 9 sec.?
4. There are 10 eggs to 6 table-spoons of milk in a pancake recipe. How many table-spoons of milk are there to 15 eggs?
5. In 20 seconds, a computer printer prints 15 lines. How many lines does it print in 8 seconds?
6. 4 identical pumps empty a swimming pool in 40 hours. How long would it take 20 pumps to do so?
7. 5 identical lorries remove a heap of rubble each by driving 12 times. How many would each of 15 lorries have to drive to remove the same heap?
8. A water supply lasts for 6 days if you daily take 18 liters. How many liters may be taken daily, if the water supply is to last for 4 days?
9. For 8 sheep, a feed supply lasts 15 days. How long would the same feed supply last for 12 sheep?
10. A certain amount of potatoes is filled into 15 kilogram-bags. 8 bags are filled. How many 6 kilogram-bags could have been filled using the same amount of potatoes?

According to preliminary examinations and to investigations by Lybeck (1978), Karplus (1983), Noelting (1980), Vergnaud (1983), pupils' successful strategies can roughly be classified into the two following forms, called the "A-Form" and "B-Form" by Lybeck:

A-Form: The children try to first establish a multiplicative relation between a and b (according to the proportional and the inversely proportional coefficient respectively) and then transfer it to the pair (c, x).

(This strategy is called "Within strategy" by Noelting , and "Function" by Vergnaud).

B-Form: The children try to first establish a multiplicative relation between a and c (according to a scalar operator) and then transfer it-or else the reciprocal operator with inversely proportions-to the pair (b, x).

(This strategy is called "Between-strategy" by Noelting , and "Scalar" by Vergnaud).

Remarkable types of errors are:

Additive strategies (add): The pupils try to establish a relation between a and c (similar to the B-Form) but chose an additional operator and transfer it to the pair (b, x).

Dividend and divisor exchanged (div): The pupils exchange dividend and divisor where a division is required.

Wrong type of function (wf): The pupils take a proportional problem for an inversely proportional one and vice versa.

No attempt made (na): No attempt is made to solve the problem.

Results of the preliminary (p) and of the follow-up test (f) (data shown in percentage referring to the total number of pupils (N = 217):

Proportions:

No. of evaluation	1		2		3		4		5	
	p	f	p	f	p	f	p	f	p	f
Success rate	79	85	60	81	69	80	34	67	15	60
A-form	28	28	4	18	65	60	5	21	5	25
B-form	50	55	53	61	4	17	23	37	8	26
ad	1	-	4	-	7	3	18	3	13	3
div	-	1	2	5	-	4	10	13	14	12
wf	-	3	-	2	1	5	3	3	2	6
n a	3	4	8	3	5	2	20	6	28	7

Preliminary test:

On the one hand, it becomes clear that with increasing "unfavourable" ratios the success rate in solving the problem is decreasing and the application of additive strategies as well as not attempting the problem is rising.

No. 2 and 3 show clearly that the children look for integral ratios when chosing their strategies. No. 1 shows that given an integer A- and B-ratio, pupils prefer the B-strategy.

Most of the B-strategies in excercise 4 (21 %) also show that pupils are inclined to use correct additive strategies: The ratio $c:a = 3:2$ allows the application of the special strategy $f(c) = f(a+a/2) = f(a)+f(a)/2 = b+b/2$ which utilizes the additivity of the proportional function f.

The interviews have shown that the successful application of the B-form in no. 1 and 2, too, is based on the concept of multiplication as a short form of writing an addition. The children try to find out, how many times the magnitude a goes into b, get the scalar operator, and transpose it in M_2, or they even add a+a+... until they get to b, then count the number of times they have added a and so add c+c+...

If this concept is no longer applicable, pupils switch to the wrong strategy "add", which only reflects the monotonous character of the proportionality.

Difficulties occur when pupils try to apply the A-form in no. 4 and 5: here, too, a confined concept of multiplication and division becomes clear manifesting itself according to the following rule which was valid during elementary school education: "You can only divide the larger number by the smaller one".

Fischbein (1985) showed, that it is difficult for pupils to detach themselves from these "Implicit Primitive Models".

Follow-up test:

The still high percentage of "div"-mistakes in no. 4 and 5 is due to pupils' failure who learnt a procedure based on the A-strategy. "Implicit Models" of division could not be reduced in this case.

The type of error "add" hardly occurs because the pupils rarely did any addition or subtraction during the teaching unit.

Results of the preliminary (p) and of the follow-up test (f):

Inversely proportions:

No. of evaluation	6		7		8		9		10	
	p	f	p	f	p	f	p	f	p	f
Success rate	39	66	54	69	38	47	26	55	56	60
A-form	10	32	28	35	36	35	22	39	54	49
B-form	26	24	25	33	2	9	3	15	2	9
ad	2	-	4	1	5	1	20	4	4	2
div	-	-	-	-	-	-	-	-	-	-
wf	8	14	9	11	27	42	9	18	2	21
n a	4	3	13	7	11	5	27	11	27	10

Preliminary test:

There is no tendency towards a decrease in the success rate with increasing "unfavourable" ratios of numbers. This was not expected anyway because the A-strategy - in this case beginning with a multiplication which is followed by a division with an integer result - is always a strategy that avoids fractional numbers. Yet, pupils like to use the B-strategy even with an integer B-ratio (no. 6,7). "Correct" additive strategies as used in proportions (f(x+x+...+x) = f(x)+f(x)+...f(x)) do not occur in inversely proportions. The fact that pupils still try to use them, partly explains why the percentage of "wf"-mistakes with inversely proportions is higher than with proportions. To what extent the problems for pupils are influenced by the ratios, is indicated by the extremely high percentage of the "wf"-mistake in no. 8, for, here, the integer A-ratio (b:a = 18:6) provokes a proportional A-strategy. In order to exclude the possibility that other factors - e. g. text variables - caused the mistake, the texts of no. 8 and 10 were exchanged by keeping the numbers in another investigation. The results were similar.

In no. 9, the high percentage of "add"-mistakes with low percentage of A-strategies at the same time, is caused by the presented situation. The intermediate result a·b belonging to the A-strategy is more difficult to interpret in this problem than in the other four inversely proportional problems. Switching to a B-strategy - with "unfavourable" ratios of numbers given - leads to the "add"-mistake.

Follow-up test:
The most remarkable result of this test is the distinct increase in "wf"-mistakes when compared to the preliminary test. Based on the interviews, we are able to name an important factor for this result: In the preliminary test, no pupil has as yet determined a pattern to solve the problems but each new problem requires pupils to find a way to solve it, i. e. the pupil has to form hypotheses for his solving process from the concrete context of the problem, to calculate and interpret partial results thus, to test his hypotheses and thereby to solve the problem sequentially and within close analysis of the concrete context.
This situation differs considerably from the one in the follow-up test: The pupil has learnt to relate the excercises with the concepts "proportion" and "inversely proportion". After having decided on the type of function he is now capable of using the respective procedure mechanically like a computer programme. His input, i. e. his analysis of the context of the problem, is confined to the decision on the type of function. Especially here lies the danger.
Let us have a look at no. 8 for example:

> A water supply lasts for 6 days if you daily take 18 liters. How many liters may be taken daily, if the water supply is to last for 4 days?

In many interviews, the type of function was determined wrongly: "The more days, the more water will be used. That is: proportional"
There is, indeed, a proportional relation - as just mentioned - in the first clause of many inversely proportional problems. Pupils chose their procedure on the basis of this relation. The procedure then runs without further text analysis and often, too, without applying the final result to the content once again. Thus, there is no protection against a wrongly determined type of function.
The previous explantations have shown possible dangers of automating the solving of word problems to a large extent.

That part of the solution which cannot be automated is reduced to a minimum. This is a correct and possible way to solve word-problems, but for the children, it is a new and unfamiliar way to work with concepts like "function", "proportions", "inversely proportions" and to solve problems by applying procedures which are based on characteristics of these types of function.
Obviously, teaching does not take this aspect into account sufficiently.

References

Fischbein, E. et al.: The role of implicit models in solving verbal problems in multiplication and division. In: Journal for Research in Mathematics Education (1985),16
Hart, K.: Childrens understandig of mathematics: ratio and proportion. London 1981
Karplus, R. et al.: Proportional reasoning of early adolescents. In: Lesh, R. & Landau, M.: Acquisiton of Mathematics Concepts and Processes. New York 1983
Kurth, W.: Problem solving with schematic procedures illustrated by word problems based on proportional and inversely proportional functions.
In: Bergeron, J., Herscovics, N. & Kieran, C.: Proceedings of the Eleventh International Conference Psychologie of Mathematics Education, Vol. II, Montreal 1987
Lybeck, L.: Studies of mathematics in teaching of science in Goteborg. Goteborg, Institute of Education, 1978, No. 72
Noelting, G.: The development of proportional reasoning and the ratio concept: Part II - Problem structure at successive stages; problem solving strategies and the mecauism of adaptive restracturing. In: Educational Studies in Mathematics, 1980, 11.
Vergnaud, G.: Multiplicative structures. In: Lesh, R. & Landau, M.: Acquisiton of Mathematics Concepts and Processes. New York 1983

CONSTRUCTING BRIDGES FROM INTRINSIC TO CARTESIAN GEOMETRY

Chronis Kynigos
Institute of Education University of London

Abstract. Turtle geometry, apart from being defined as intrinsic, has a special characteristic; it invites children to identify with the turtle and thus form a body syntonic thinking "schema", to drive it on the screen to make figures and shapes. This is a report of on - going case study research, whose aim is to investigate the potential use of this "schema" by young children in order to develop understandings of Cartesian geometry. Three pairs of 11 -12 year old children with 50 -60 hours of experience with turtle geometry participated in the study.The results presented here highlight the children's conflicts arising from their attempts to use a coordinate method to control the turtle. A model of a synthesis of their insights into coordinate notions is proposed, together with a model of the schema they seem to have built during their experience with turtle geometry prior to the study. Some examples are then given of the children's dissociations from their "intrinsic schema" and their subsequent understandings of specific coordinate notions.

The theoretical framework of the study is based on the role of Logo and turtle geometry within a specific view of mathematics education; i.e. learning mathematics is seen as an on - going re - organisation of personal experience, rather than an effort to describe some ontological reality. The child learns mathematics by building with elements which it can find in its own experience (Von Glaserfeld, 1984). Papert (1972) uses words like "doing" and "owning" mathematics to stress the dynamic and active involvement of the child. Hoyles and Noss (1987) use the notion of "functional mathematical activity", i.e. the child using mathematical ideas and concepts as tools to solve problems in situations which are personally meaningful. Logo is seen by more and more educators as a powerful tool for creating educational environments in accordance with the above perspective. Turtle geometry, a very important part of Logo, has a particular characteristic; when children "do" turtle geometry, they can identify with the turtle, and therefore use personal experience of bodily motion to think about the shapes and figures they want to make (Papert 1980, Lawler 1985).

My approach to this thinking "schema" which the children seem to adopt for doing turtle geometry (called "intrinsic thinking" by Papert and Lawler), does not pre - assume the nature of the geometrical notions used when the schema is employed to drive the turtle on the screen. It is infomed, rather, by research into the structuring of intuitive geometrical knowledge, i.e. the way children link very simple sets or "units" of such knowledge to the turtle's actions. They acquire these "units" from very early personal experience of movement in space. DiSessa might call these units "phenomenological primitives", although his study was in the context of physics (diSessa 1982). Lawler puts forward the notion of a "microview" to talk about domain specific fragments of personal experience. He contends that the personal geometry "microview" is "ancestral" to the intrinsic geometry "microview".

However, the nature of the geometrical notions underlying turtle geometry is characterised by Papert as intrinsic, i.e. that turtle geometry belongs to the family of the differential geometrical systems where growth is described by what happens at the growing tip (Papert 1980). This geometry is contrasted to the "logical" euclidean geometry of theorems and proofs and to the "analytical" cartesian geometry where changes of state are caused by location descriptions. Papert discusses the different nature of these geometrical systems and argues that it is mathematically important for children to understand the relations between them. As part of a wider issue of the potential of intrinsic thinking for the learning of geometry (Kynigos 1987a), this study addresses the problem of whether it is possible for children to use this powerful thinking tool which they adopt naturally from doing turtle geometry, to develop an understanding of the cartesian geometrical system and its relationship with the intrinsic. For convenience this thinking schema will be refered to as the "intrinsic schema".

OBJECTIVES.

The aim of the study was to investigate in detail different aspects of the same problem i.e. the extent to which it is possible for children to use their intrinsic schema for developing an understanding of coordinate geometry. The method employed involved the ecouraging of the development of three separate learning paths, each employing a different conceptual base for describing the plane (fig.1), thus building a different "bridge" from intrinsic to cartesian geometry. All the paths consisted of three categories of activities (fig. 1) with the aim of:

Category 1) illuminating the process by which the children developed an understanding of a systematic description of the plane (fig. 1 - A, B, C).

Category 2) illuminating the nature of children's understandings of the absolute coordinate and heading systems, while using a non - intrinsic method to change the turtle state in the coordinate plane (fig. 1 - D, E).

Category 3) investigating if and how they used their intrinsic schema to relate intrinsic and coordinate notions while choosing a method of changing the turtle state in order to make measurements on the coordinate plane ("T.C.P." microworld, fig. 1, F).

METHOD.

Three pairs of 12 year - old children participated in the study, one for each path (fig. 1). Prior to the study the children had had around 60 hours of experience with turtle geometry (they did not know about the SET- commands) in an informal, investigation - type classroom setting as members of a Logo club of 20 children in total. The research was carried out during school hours in a small "research room" and each pair of children participated in three 90 to 120 minute sessions, one for each category of activities, in a total period of no more than a week for each pair. Soft and hard copies were produced of everything that the children said and typed. The researcher also kept what

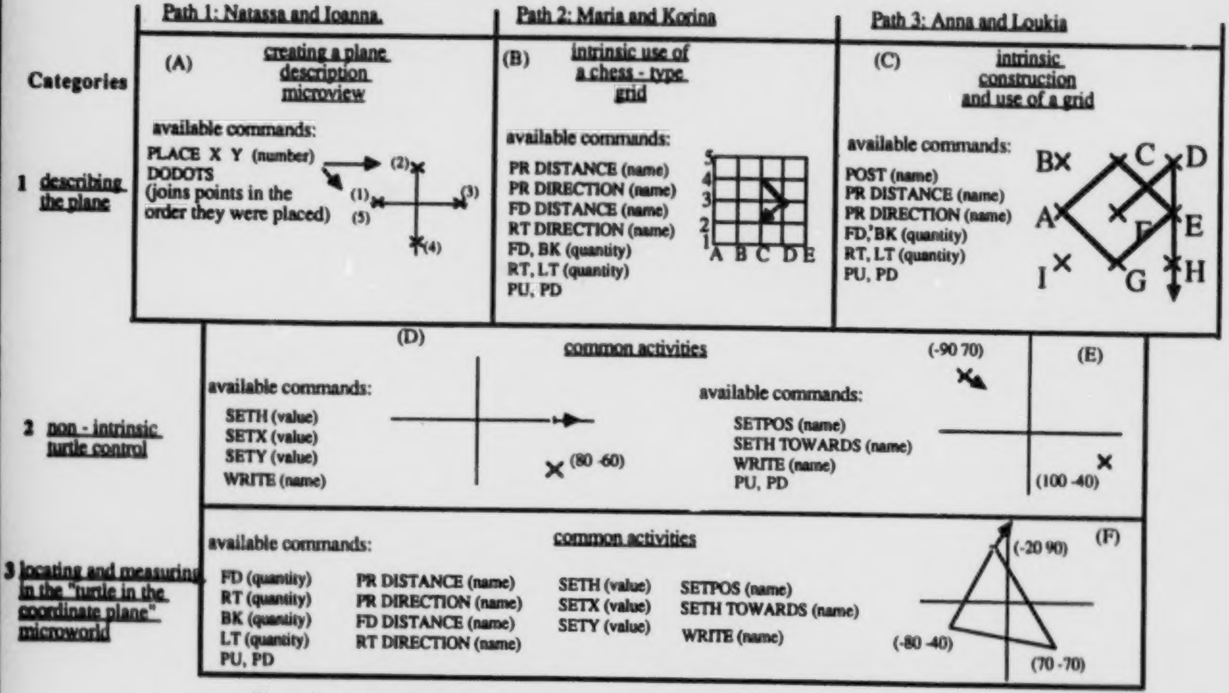

Path 1: Natassa and Ioanna.
Path 2: Maria and Korina
Path 3: Anna and Loukia
Categories
1 describing the plane
(A) creating a plane description microview
available commands:
PLACE X Y (number)
DODOTS
(joins points in the order they were placed)
(1)
(2)
(3)
(4)
(5)
(B) intrinsic use of a chess - type grid
available commands:
PR DISTANCE (name)
PR DIRECTION (name)
FD DISTANCE (name)
RT DIRECTION (name)
FD, BK (quantity)
RT, LT (quantity)
PU, PD
1
2
3
4
5
A B C D E
(C) intrinsic construction and use of a grid
available commands:
POST (name)
PR DISTANCE (name)
PR DIRECTION (name)
FD, BK (quantity)
RT, LT (quantity)
PU, PD
A
B
C
D
E
F
G
H
I
2 non - intrinsic turtle control
(D) common activities
available commands:
SETH (value)
SETX (value)
SETY (value)
WRITE (name)
(80 -60)
available commands:
SETPOS (name)
SETH TOWARDS (name)
WRITE (name)
PU, PD
(-90 70)
(E)
(100 -40)
3 locating and measuring in the "turtle in the coordinate plane" microworld
available commands:
common activities
FD (quantity)
RT (quantity)
BK (quantity)
LT (quantity)
PU, PD
PR DISTANCE (name)
PR DIRECTION (name)
FD DISTANCE (name)
RT DIRECTION (name)
SETH (value)
SETX (value)
SETY (value)
SETPOS (name)
SETH TOWARDS (name)
WRITE (name)
(F)
(-20 90)
(-80 -40)
(70 -70)

Figure 1: A diagram of the structure of the study and the tasks involved in each category of activities.

BEST COPY AVAILABLE

they wrote on paper, produced graphics screen - dumps, and took notes on anything of importance which would escape the rest of the data.

A rather detailed analysis of the data was required in order to understand the children's thinking processes, especially at times of conflict created by environments which embedded notions which were "dissonant" to their hitherto experience. A substantial component of the analysis therefore is in the form of "significant" episodes during the children's activities illustrating the nature of their insights or confusions related to the research issues.

RESULTS.

The results presented here concentrate on the activities of the children during the category 2 tasks which involved taking the turtle to specific points on the coordinate plane (shown on the screen by a cross sign) with the only available means being the coordinate (SET) commands (fig. 1, D). For the category 2 and 3 activities (fig. 1 D, E, F), the researcher imposed position changes dependent on the turtle's heading, i.e. the turtle could only move towards where it was heading.

As a result of the analysis of the data, a model of a "coordinate schema" is being developed, which synthesises the children's insights into the notions involved in the coordinate controlling of the turtle. The model consists of heading and position change schemas, which the children seemed to be in the process of building as a result of dissociating from intrinsic notions. This process of dissociating from the intrinsic schema and developing another, seemed to throw light on specific notions the children had aparently built for controlling the turtle during their 15 - month experience with turtle geometry, thus clarifying components of the intrinsic schema itself, a model of which is also proposed in the study.

During an earlier part of the analysis (Kynigos 1987, b), examples were given of one pair of children (pair 3, fig.1) dissociating from a turtle "action - quantity" schema (e.g. "move steps, turn degrees") while having insights into important factors for changing the heading and the position in the coordinate plane. This report presents examples of how the other two pairs of children seemed to make dissociations from "action - quantity" and sequentiality notions in "intrinsic" heading and position changes in order to solve the tasks. The presentation concentrates on the children's understandings of a turtle state - change caused by describing the end state (e.g. the meaning of the command SETH 180), rather than a "sequential" change from the present to the end state (e.g. the meaning of RT 180). The children's abandonning of their "action - quantity" schema is also illustrated in favour of state changes caused by descriptions of absolute directions and locations.

The following episode illustrates Maria and Korina's first insight into the coordinate method of changing the turtle's heading which seemed to involve a dissociation from their familiar "action -

quantity" schema and the use of an absolute direction system to describe the new heading. The first discussion concerning the method of changing the turtle's heading arose in the context of a mistake during the task to take the turtle at point 80 -60, i.e. Maria's aparent unclear distinction between the two states and the nature of their metric systems (degrees and length units), resulting in her typing in SETH and then counting on the x - axis for an 80 input to SETH (point at 80 -60). The process of discussing the meaning of the SETH command and its input in order to understand the turtle's resulting heading of 80, seemed to favour the development of an awareness of an external direction as the determinant of heading change. The following extract illustrates the aparent carry - over of this awareness to the next task (turtle at -90 0, heading 270, point at -90 -40, fig. 2):

(discussion on how to take the turtle from heading (1) to heading (2))

M: "SETH..."
K: "To show where it's
looking, yes..." (meaning of SETH)
M: "SETH..."
K: "How much...wait...
to look downwards..." (meaning of the input)
M: "SETH 180."

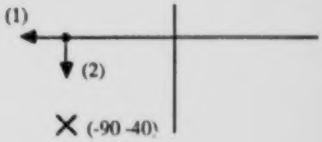


Figure 2
M. and K.: Discussing the meaning of SETH

However, it seems that this insight in dissociating heading change from action - quantity, did not incorporate a dissociation of what has been referred to as the "sequentiality schema", i.e. the notion the children seem to have built from their turtle geometry experience, that a heading change is caused by a turtle action from its previous heading to the new one.This can be illustrated by the children's attempt to take the turtle on the -100 90 point (fig. 3), a task in which the axes were hidden. Having passed the point by typing SETY 100, the children were trying to make the turtle face downwards, i.e. change its heading from 0 to 180. Although Maria's verbal expression of her plan seemed to indicate an understanding of relating heading change to an absolute direction ("...this is 0 now, if we turn and we say SETH 180..."), she had not really seen the absolute direction as the only necessary determinant of the change. This became aparent in her attempt to make the turtle face downwards from a heading of -20 (she had typed SETH -20 confusing degrees with turtle steps - fig. 3)

(Discussing how to change the turtle's heading from (1) to (2))

M: "So we should tell it to go to 180.
Therefore, 200. Let's see..." (she types SETH 200))

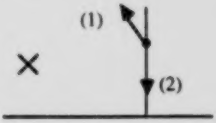


Figure 3
M. and K.: Discussing how to make the turtle face downwards

It is suggested that Maria's mind focussed on the rotational "distance" from -20 degrees to 180, imposing an input which was dependent on the previous heading. This sequentiality schema seemed to have a very strong resistance to change in the children's mind; after discussing the outcome and trying out different inputs to SETH, Maria did seem to have an insight into the absolute nature of this method of heading change:

M: "I.e. however much it is, let's say 5 degrees further, it's not relevant, let's say we mustn't add it to..."
K: "We should put it normally (she means just the end heading) whatever it is."
M: "Good. Now let's tell her... 10 distance."

Inspite of the different context (change of position) it was seen as important to include the last phrase of this dialogue, which seems to indicate that although Maria had just had an insight into the notion of end direction being the important factor in changing the heading, she did not carry that notion to the change of the turtle's position from (0 100) to (0 90), focussing on the distance from 100 to 90. In fact, the children had already discussed changing the position before turning the turtle, imposing a distance notion in their plan (fig. 4):

(Discussing how to take the turtle from position (1) to position (2))

M: "No, it's too much."
K: "Yes... a bit less."
M: "Em... minus 10. Minus 20, therefore 80."
K: "Yes, I said 80 at the beginning too."
M: "O.K., -20 then."

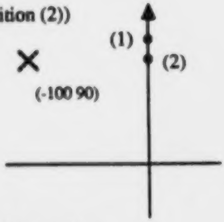


Figure 4
M. and K.: Changing the turtle's position

The children seemed to be talking about the turtle steps from the 100 to the 80 point, i.e. the distance from the present position to the position of change. They also seemed to impose a "reverse action" notion, of "undoing" an aparent forward 100 action by subtracting the distance.

The strength of this "relative distance" (as opposed to distance from the origin) schema is illustrated by the children's persistence to use it in their subsequent activities: at first they typed in -20, forgetting about the SETY command. After discussing the error message from the SETY -20 command, which led to a turning of the turtle to face downwards, and although Maria had had an insight into the notion of the end direction being the important factor in changing the heading (fig. 2), she did not carry that notion to the change of the turtle's position from (0 100) to (0 90). Focussing on the distance from 100 to 90, she typed in SETY 10, and after the result on the screen, SETY -10, aparently thinking she had failed to include a "reverse action" element.

From the resulting 0 -10 position, the children turned the turtle to face upwards again and took it to (0 80), saying forward 80 and typing SETY 80. Only then, did one of them (Korina) show some indication of dissociating from the relative distance notion, expressing an opposition to a proposed SETY 10 command in the attempt to move from 0 80 to 0 90:

M: "Now. SET...Y... 10."
K: "10? I say, let's do... 90."

However, the children did not explicitly use the notion of position change caused by giving the end position as an input, in any of the subsequent tasks in this session.

Natassa and Ioanna, however, were more explicit in their attempts to make sense of position changes. They met their first difficulties in trying to move the turtle from a -100 0 to a -110 0 position in order to decide whether the value of 100 for the x coordinate was the correct one (the axes were invisible, the point was at -100 90, fig. 5). In their efforts to explain why their first attempt (SETX -10) did not work while their second (SETX -110) did, the children constructed a "theory" for the meaning of the number of the x value.

(explaining why SETH -110 worked while SETH -10 did not,
in taking the turtle from position (1) to position (2))

I: "...we did it again from 0 till 110
and it came out."
N: "...we can't do 10 because we've
done 100 already. Plus 10 we wan
to do... 110."
I: "She doesn't go... because we've
passed 10."

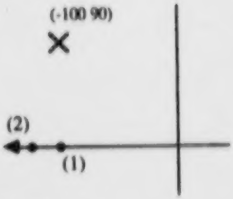


Figure 5
N. and I.: Changing the turtle's position

Ioanna seemed to suggest two ways of interpreting the meaning of the x value: firstly, the value represents the distance from the origin, and therefore the SETX command operates in such distances, and secondly it represents a name for a place ("...we've past - the place - 10."). Natassa seemed to take on board the "distance from the origin" theory. Notice how she used a specific way to talk about a number when it represented an x value (by using the word "do" in front of such numbers), and seemed to implicitly contrast it to the normal meaning of number ("... plus 10, we want to do... 110).

DISCUSSION.

In their attempts to control the turtle, the children seemed to dissociate from their intrinsic schema and develop new schemas for heading and position changes. Not surprisingly, this development was not uniform across pairs, or across children individually. The children seemed to have "insights" into parts of the coordinate method at various times during the activities but no child seemed to explicitly synthesise the notions into a concise method of state change. The model for the "coordinate schema" which is being developed, therefore, is only a synthesis of the children's insights into the notions involved in controlling the turtle in the coordinate plane.

The study provides a description of the process by which the children aparently began to build a mental schema with dynamic characteristics, i.e. one which would enable them to make controlled changes in the coordinate plane. The schema seemed to emerge in the children's minds from it's antithesis to the intrinsic schema, caused by the coordinate nature of the category 2 tasks (fig. 1). It is interesting to consider the relationship between these two schemas and in particular that they both seemed to emerge (at different times) in the children's minds as mental tools for making changes in particular environments. Although this does not come as a surprise in a turtle geometric environment, it is not a self - evident characteristic of the learning of coordinate geometry. In the category 3 activities, for example, where the children could choose the method for controlling the turtle (fig. 1), they seemed to use the necessary coordinate notions (e.g. locating and naming methods) either by employing their intrinsic schema (e.g. FD DISTANCE 70 -70: "go forward the distance from where you are to point 70 -70"), or their coordinate schema (e.g. SETPOS 70 -70: "put yourself on point 70 -70"). It seems therefore interesting to consider the potential of the "T.C.P." microworld of the category 3 activities (fig. 1) in providing the children with the opportunity of a dynamic interplay between the two geometrical systems by means of the option to employ a method to make changes, based on concepts belonging to either system.

REFERENCES

diSessa, A., (1983) Phenomenology and the Evolution of Intuition. In Gentner and Stevens, editors, Mental Models. N.J. Lawrence Erlbraum

Hoyles C. and Noss R. (1987) Children working in a structured Logo environment: from doing to understanding. Recherches en Didactiques de Mathematiques, vol 7.

Kynigos, C. (1987 a) From Intrinsic to non - Intrinsic Geometry, Proceedings of PME 11, vol.3, Montreal, ed. Bergeron, J. Herscovics, N. Kieran, K.

Kynigos, C. (1987, b) A Case - Stydy of two 12 year - old children changing the turtle's state in the Coordinate Plane, Proceedings of the third international conference for Logo and Math. Educ., Department of Mathematics Concordia University, Montreal

Lawler, R. (1985) Computer experience and cognitive development, Ellis Horwood.

Loethe, H. (1985) Geometrical Problems for a Turtle with Direction and Distance finder, Proceedings of the first Logo and Math. Educ. Conf. Univ. of London, Inst. of Ed.

Papert, S. (1980) Mindstorms, Harvester press.

Papert, S. (1972) Teaching Children to be Mathematicians versus Teaching about Mathematics. Int. J. Math. Educ. Sci. Technol., 3.

CONCEPTS IN SECONDARY MATHEMATICS IN BOTSWANA

Hilda Lea
University of Botswana

Abstract

Four tests from "Concepts in Secondary Mathematics and Science" tests were used on a sample of secondary school pupils in Botswana. The aims were to ascertain the levels of understanding of pupils in Forms 2, 3 and 4 and to identify difficulties; to compare performances of boys and girls in Botswana; and to attempt to make some comparison with available results for some questions on the same tests carried out in England. It was found that many pupils were still at the concrete operations stage in secondary school; that there was a small difference between performance of girls and boys, with a trend in favour of boys; and that there was some relationship between the performance of pupils in Botswana and England.

INTRODUCTION

The CSMS tests were very carefully constructed, with questions chosen to examine a variety of concepts in an unfamiliar setting, so that hierarchies of understanding could be established, and an investigation of children's difficulties made. This should give insight into the way children learn mathematics Hart (1981). Levels of questions linked to Piaget's stage theory, were used as a framework to describe pupil's understanding. Four levels were identified. Level 1 shows an understanding of basic concepts. Level 2 shows the application of these concepts. Level 3 shows the beginning of abstraction. Level 4 uses abstract reasoning as well as the application of knowledge to the solution of problems. In Piagetian terms it could be said that items at Levels 1 and 2 require early concrete operational thought, Level 3 late concrete, and Level 4 early formal operational thought.

Achievement Of Girls And Boys

In Botswana, girls constitute 60% of the junior secondary school population. Form 4 is selective and girls constitute 40% of the senior secondary school enrolment. This suggests that boys are already showing greater ability. Boys perform better in O level mathematics and more boys achieve higher

placings in the junior and senior maths contests. Kahn (1981) showed that in Botswana, educational achievement due to sex difference is not significant at primary level but is more pronounced at Form 3, and that boys perform better than girls in all subjects except Setswana.
In Britain, APU (1980) showed that even though all girls do mathematics up to 16 years, they are not as successful as boys, and only 39% of the top 10% in O level were girls. Boys were shown to be ahead in descriptive geometry. In USA, NAEP (1980) found that at 14 years of age boys and girls did equally well, but at 17 years fewer girls study maths and those who do have a lower achievement. Fennema (1978) found that boys were better at spatial visualisation. The generalisation of many studies in Russia also showed more mathematical ability for boys. Schildkemp-Kundiger (1982) on an international maths study, found some sex related differences in maths achievement in a wide range of countries of different economic levels.

Comparison Of Results In Botswana And England

As mathematics is more culture free than most subjects, it should be possible to make some comparisons. It should be borne in mind that in Botswana, the medium of instruction from Standard 4 in primary school is English; in Botswana the JC examination is taken at the end of year 3, whereas in England there is no national examination till the end of year 5; and in both countries O level is taken by the most able, approximately 11% in Botswana and 20% in England.

METHOD

Sample

Ten schools were chosen at random, and 15 boys and 15 girls were selected at random from each Form used. At the time of the tests about 35% of the primary school population went on to secondary school and approximately 11% of the primary intake proceeded to Form 4. In England the sample was mostly taken from Comprehensive schools with a large ability range. A quarter of each year group was used, the sample being representative of the normal distribution of IQ in the British child population. The population of England to Botswana is in the ratio 50 : 1, and though the sample in England is larger, the Botswana sample covered a much greater percentage of the school population.

Procedure

Forms 2, 3 and 4 took the same tests in October each year. Papers were returned to the University for marking and analysis. Answers were coded, 1 for correct answers, 0 for completely wrong or missing answers, and 2-9 depending on the type of mistake made. The marking system was that shown in Hart and Johnson (1980).

RESULTS

Table 1

Summary of results showing the percentage of pupils giving correct answers.

		Botswana			England		
		Form 2	Form 3	Form 4	Form 2	Form 3	Form 4
Measurement	Level 1	71	71		85	86	
	2	50	56		65	76	
	3	33	41		42	48	
	4	16	16				
	n	160	126		444	373	
Reflection & Rotation	Level 1	58.2	66.3	84.8	79**	80.9	84.7
	2	58.1	59.0	83.6	52.6	57.6	64.8
	3	29.6	31.8	59.6	35.6	41.6	47.0
	4	39.0	42.6	59.9	25.1	27.4	35.6
	5	17.7	22.5	35.0	12.3	16.0	20.2
	n	105	147	66	293	449	284
Algebra	Level 1	92.5	92.5		86.7	92.5	
	2	69.7	78.6		51.7	65.4	
	3	39.9	44.9		26.5	43.8	
	4	10.8	18.4		7.6	14.7	
	n	221	176		1128	961	
Graphs	Level 1	59.5	55.8	78.1	67.3	63.3	69.6
	2	40.6	37.0	59.1	17.0	18.3	21.9
	3	30.3	28.7	68.0	21.6	19.0	30.0
	n	215	115	68	459	755	584

Table 2

Results of girls and boys in Forms 2, 3 and 4 showing the percentage of pupils giving correct responses.

Test	Level	Form 2		Form 3		Form 4	
		Girls	Boys	Girls	Boys	Girls	Boys
Measurement	1	66	77	70	74		
	2	45	57	53	63		
	3	27	39	37	44		
	4	20	23	23	22		
	n	85	75	70	56		
Reflection & Rotation	1	52	66	61	72	81	90
	2	44	53	41	58	78	79
	3	20	40*	23	42	63	62
	4	31	47	34	47	52	67
	5	9	25*	11	19	30	36
	n	54	53	56	36	27	39
Graphs	1	67	67	69	50	90	72
	2	35	44	34	36	57	61
	3	19	28	22	23	58	62
	n	85	62	68	47	21	47

*Significant at .05 level

DISCUSSION OF RESULTS

When the tests were drawn up, items were identified which had the same level of difficulty. If 2/3 of the pupils answered an item correctly, it was an indication of the level reached by the group. Similarly a pupil answering 2/3 of all questions correctly at a particular level, would be considered to have reached that level of ability. From Table 1 it would seem that in general in secondary schools in Botswana, in Form 2, 22% give responses classified as early formal, 11% late concrete and 37% early concrete; in Form 3, 26% give responses classified as early formal, 11% late concrete and 35% early concrete; in Form 4, 35% give responses classified as late formal, 25% early formal and 4% late concrete. In most cases Form 3 results were better than Form 2. In the graphs test Form 2 results were better, probably due to the fact that this topic had just been completed in the syllabus. From Table 2, one can compare samples of girls and boys. There is a consistent trend in favour of boys, significant in two cases in the Reflection and Rotation test in Form 2. This would support research findings that in maths boys perform better in general, and noticeably better in spatial visualisation. It is not possible to make precise comparison between samples in Botswana and England though trends can be identified. It must be remembered that this is a comparison between the top third of school age children in Form 2 and 3 in Botswana, and the whole ability range in England. Selective Form 4 in Botswana is being compared with an all ability range in England.

Table 1 shows that in measurement, the sample of pupils in England did noticeably better at levels 1, 2 and 3. This could be partly due to the fact that the environment of many pupils in Botswana is frequently unstructured. In Reflection and Rotation, pupils in England did better at levels 1 and 3, and pupils in Botswana performed better at levels 2, 4 and 5. In Algebra pupils in Botswana performed better at all levels. In Graphs, pupils in England did better at level 1, but pupils in Botswana did noticeably better at levels 2 and 3. The fact that Botswana performance is relatively poorer at level 1 but relatively better at other levels, suggests that in England pupils get a better foundation in mathematics at primary school, but at secondary school performace on average deteriorates, in comparison to Botswana where the opposite seems generally to be true. This is supported by the fact that pupils in Botswana did better in Algebra at level 1, a subject not done at primary level. It is not possible to make a fair comparison between Form 4 results. One can only say that the average performance of Form 4 pupils in Botswana (11%) is better than the average performance of all Form 4 pupils in England.

CONCLUSION

Considering that at Independence in 1966, very few pupils in Botswana received secondary education, then it is clear that enormous strides have been made since then. At present 35% of the possible school population take the Junior Certificate examination at the end of year 3, with over 70%

pass rate. In O level, results have consistently improved since 1979 with a pass rate of 55%, to a present pass rate of over 80%. The overall improvement in results is probably due to two main factors - that there are many more good Batswana mathematics teachers in the schools, and that the text books widely used were written for Botswana, Lesotho and Swaziland by local teachers and mathematics educators. Overall, evidence suggests that, in Botswana, many pupils have difficulty with formal reasoning well into junior secondary school. The implication for teaching is that, if pupils can only function effectively at the concrete operations stage, materials must be presented in a way which is directly related to everyday situations, otherwise they may be reduced to learning algorithms with little understanding. Many senior secondary pupils do not function consistently at the level of formal reasoning, so it is important to relate some of that work also to everyday situations.

In a mixed ability class pupils will be at different stages in making the transition from concrete to formal operations, so weaker pupils may not yet be able to do questions with very abstract reasoning, yet more difficult questions must be given to the better pupils if they are to reach their full potential. Results suggest that boys do perform better than girls in mathematics. That this should be so in Botswana is interesting, because in a country as new as this there is not likely to be a tradition of stereotyping related to role or to subject. Social and cultural factors could play some part. There are also possible explanations in terms of brain laterality, genetic or hormonal influences. One aspect of the research was to ascertain whether pupils in Botswana performed very differently from those in England. It was found that on the whole they did not. They made the same types of errors and levels of cognition measured by the achievement of the pupils, was comparable. It would be interesting to compare results in CSMS tests with those from other countries.

REFERENCES

Cockcroft, W . Mathematics Counts HMSO 1982.

Hart, K. Children's Understanding of Mathematics 11-16 Murray 1981

Hart, K. & Johnson, D. A report of the mathematical component of the concepts in secondary mathematics and science programme, Chelsea College, London 1980.

Shayer, Kuchemann, Wylam The distribution of Piagetian stages of thinking Br. Journal Ed. Psych. 44, 2, 155-64 1976.

A DEVELOPMENTAL MODEL OF A FIRST LEVEL OF COMPETENCY IN PROCEDURAL THINKING IN LOGO : *"May be we're not expert, but we're competent"*

TAMARA LEMERISE
UNIVERSITE DU QUEBEC A MONTREAL

Abstract. The present paper addresses the problem of the kind of competence a child between 9 and 11 can develop in procedural thinking applied to structural programming in Logo. A present trend in the literature tends to demonstrate that children of this age can not master structural programming. There is some truth to that, but nontheless we can still find a body of evidence showing that they do acquired a certain competence in the domain. We propose here a model describing a *first level of competency* that can be acquired by children of this age. The proposed model tries to capture the path followed by these children in their progressive use and conceptualisation of five of the main caracteristics of the Logo procedure. The following goals may be served in presenting our model : illustrate children's habilety to develop some competence even if they do not become expert; propose some guidelines to teachers or researchers interested in the promotion of such competency; and finally argue that it is possible to obtain evidence of children's competency in this domain at age levels younger that of secondary school children.

Dans le domaine des recherches et applications Logo on observe présentement une dérive de l'intérêt vers le niveau secondaire (12 ans et plus). Durant les sept dernières années de la décennie 80, Logo s'est laborieusement taillé une place au niveau primaire (6 à 12 ans). Plusieurs chercheurs et enseignants ont déployé de grands efforts pour introduire philosophie et langage Logo dans les classes du 2 ième cycle du primaire (4-5-6 ième année) et même, à l'occasion, dans celles du premier cycle (1-2-3 ième année). Nombre d'études ont tenté soit d'analyser l'impact du travail en Logo sur le développement des enfants, soit d'en décrire les conditions idéales d'application. Aujourd'hui, c'est l'implantation et l'évaluation du Logo au secondaire qui semble le plus retenir l'intérêt des chercheurs et des professeurs.

A priori, ce phénomène est réconfortant et dans la logique même des événements. En effet, suite à la fascinante période de la diffusion du Logo dans les classes du primaire, le moment est venu, si l'on veut s'assurer du maintien et du développement de Logo dans le système éducatif, de poursuivre les efforts d'implantation aux niveaux scolaires plus avancés. Il y a certes eu de tous temps des chercheurs et pédagogues qui ont oeuvré en Logo au secondaire (à titre d'exemples l'équipe de C. Hoyles et R. Noss en Angleterre, celle de J. Olive aux Etats-Unis, ou encore celle de A. Rouchier en France); le mouvement n'est donc pas nouveau en soi, mais seulement plus accentué aujourd'hui. Il faut toutefois souhaiter que l'intérêt accru pour les niveaux d'âge plus avancés (12 ans et plus) ne soit pas associé à un désintéressement ou à une "dévalorisation" des compétences susceptibles d'être acquises en Logo par les enfants du primaire.

Que ces enfants placés dans un environnement Logo développent toutes sortes de petites compétences (apprentissage d'un langage informatique, perception nouvelle de l'erreur, vision dynamique du concept d'angle, calcul et mise en relation de dimensions, ect...) est aujourd'hui généralement confirmé et accepté. Toutefois, relativement aux *grands canons* de Logo, tels la maitrise de la programmation structurée, la compréhension de la notion de variable ou la manipulation de la récursion, les progrès observés chez cette population se sont avérés plutôt minces comparativement aux attentes (Kurland et al 1987, Blouin, Lemoyne 1987, Hillel 1984). En effet, exception faite de la récursion généralement reconnue difficile d'accès, certains espoirs ont été entretenus relativement à la compétence des 9-11 ans à maitriser la programmation structurée et à manipuler la variable dans certaines situations peu complexes. Or rares sont les enfants de niveau primaire qui, même après deux années de travail avec Logo, deviennent hautement compétents dans l'un ou l'autre de ces domaines.

Les facteurs explicatifs les plus fréquemment amenés à l'appui de ces "pauvres performances" observées chez ces enfants sont liés tantôt aux limites imposées aux conditions de travail (une heure ou deux /semaine pour des périodes de 7, 15 ou 30 semaines /année), tantôt à la nature du contexte pédagogique (l'approche "projets libres" nécessiterait un grand laps de temps avant que soit assuré un apprentissage réel et stable; l'approche "projets dirigés" s'avère souvent trop limitée pour assurer un apprentissage complet et authentique d'une habileté spécifique souvent dépendante de la maitrise d'habiletés connexes !). Un troisième facteur, plus rarement évoqué, est celui des obstacles épistémologiques créés par le type et le niveau des habiletés requises pour la réussite de la tâche. Si les habiletés requises sont fort complexes, l'enfant d'un certain niveau d'âge n'aura même pas le bagage développemental nécessaire pour pouvoir travailler à les acquérir.

Le modèle que nous voulons présenter ici se rattache à ce troisième facteur ; pour le moment, il ne couvre que la seule problématique du *développement de la compétence en programmation dite structurée*. L'argument principal que nous voulons ici promouvoir à travers la présentation de ce modèle est que la "non-compétence" observée chez les enfants de 9-11 ans, dans le domaine de la programmation structurée, est ainsi définie à cause du modèle de référence utilisé pour l'évaluer. Lorsque ce modèle de référence est un modèle d'expert, inutilement trop sophistiqué, il masque ou dévalorise toute une série de compétences locales effectivement acquises par les enfants dans leur démarche progressive menant vers la maitrise de la programmation structurée. Ce sont ces compétences "pré-expertes" ou "prérequises" que nous voulons ici dévoilées, afin de promouvoir une évaluation positive des progrès des enfants dans le domaine de la programmation structurée (malgré le fait qu'ils ne sont point encore experts), et de proposer aux chercheurs et enseignants un modèle décrivant certaines étapes importantes à franchir sur le chemin de la maitrise définitive. Les données utilisées à l'appui de ce modèle proviennent d'une part des données déjà rapportées par certains auteurs (Noss 1985, Hillel et Samurcay, 1985), mais aussi et surtout de notre propre banque de données recueillies au cours de trois années consécutives de travail avec une classe multiâge (4-5-6ième) d'une vingtaine d'enfants.

AU SEUIL DE LA PENSEE PROCEDURALE : un modèle du développement des compétences pré-requises à la programmation structurée

Dans un premier temps sera présentée une description sommaire d'une série d'habiletés en pensée procédurale, jugées ici prérequises à la maitrise de la programmation structurée. Suivra une schématisation de la trajectoire développementale de ces habiletés telle qu'observée chez des sujets âgés de 9 à 11 ans. Enfin quelques brèves recommandations sont dégagées pour la mise sur pied d'un contexte de travail favorisant le développement harmonieux, et peut-être moins laborieux, de la pensée procédurale en programmation structurée.

1-Description des habiletés

En Logo, l'habileté à programmer de façon structurée fait appel à deux grands types d'habiletés spécifiques : l'habileté à *définir* des procédures et l'habileté à *manipuler* des procédures.

L'habileté à *définir* des procédures réfère d'abord aux compétences du programmeur à sélectionner et à organiser les actions Logo jugées utiles pour reproduire une forme, un effet ou un projet donné. La mise en procédure c'est aussi l'habileté à regrouper la série d'actions choisies pour la représenter sous un seul vocable. Ainsi, la mise en procédure ne nécessite point, du moins dans un premier temps, l'élégance ou l'économie des actions choisies, mais elle requiert qu'un lien d'équivalence procédurale soit établi entre le nom de la procédure et la liste ordonnée des actions qui la composent. Que CARRE soit définie avec un répète ou par une série d'actions à la queue leu leu importe peu dans la mesure où le programmeur peut se représenter la série d'actions chaque fois évoquée par sa procédure spécifique. Aussitôt définie, une procédure est aussitôt appelée à être manipulée; ainsi l'habileté à définir des procédures appelle l'habileté à manipuler des procédures.

L'habileté à *manipuler* des procédures. En Logo, l'habileté à manipuler une procédure se manifeste habituellement sous l'une des quatre formes suivantes : 1) habileté à modifier une procédure ; 2) habileté à transformer une procédure; 3) habileté à organiser entre elles plus d'une procédures; et 4) habileté à exporter une procédure.

1) L'habileté à modifier une procédure réfère aux initiatives des programmeurs pour soit ajouter une commande, la modifier ou la retrancher dans une procédure déjà définie. Souvent, par exemple, les enfants vont ajouter de la couleur (un FCC ou un FFG) dans leur procédure, ou encore ils modifient une longueur , mieux adaptée à leur besoin du moment, ou plus simplement encore, ils corrigent, suite à un résultat inattendu, une ou des erreurs de copie. Plus tard, certains définiront directement leur procédure dans l'éditeur, sachant fort bien qu'ils peuvent la modifier si le résultat ne concorde pas à leur attente. L'habileté à modifier une procédure souligne le caractère hautement plastique de l'entité procédurale.

2) L'habileté à transformer une procédure consiste à créer une nouvelle procédure à partir d'une procédure déjà existante. La nouvelle procédure se différencie souvent de la procédure mère par des variations mineures de commandes. Par exemple, les procédures CARRE1, CARRE2 ne sont que des reprises agrandies ou rapetissées d'un CARRE précédemment défini. Dans d'autres cas, c'est un besoin de symétrie qui est à l'origine d'une transformation : telles ces procédures miroirs qui demandent d'adapter l'orientation d'un même angle (OEIL, OEILDROIT; OCTOGONE, OCTOGONEGAUCHE). La transformation d'une procédure se différencie donc de la modification d'une procédure en ce qu'elle génère une seconde procédure . Une

transformation de procédures augmente ainsi le nombre de procédures qu'un sujet a à son actif. Toutefois, il n'est point rare que la transformation donne lieu à des modifications de procédures (au niveau des noms des procédures une certaine réorganisation est souvent appliquée : CARRE1, CARRE2, CARRE3 ; OEILD, OEILG; des longueurs sont modifiées pour que les deux procédures puissent être pairées à l'écran : OCTOD, OCTOG) . Plus encore, une transformation initiale révèle à plus d'un apprenti-programmeur le jeu attrayant de la transformation "à la chaine" ; d'abord appliqué à un premier ensemble (OCTO -- OCTOD, OCTOG -- PETTITOCTOD, GRANDOCTOD, PETITOCTOG, GRANDOCTOG), ce jeu peut rapidement se généraliser (CERCLE -- CERCLED, CERCLEG, DEMI.CERCLED, DEMI.CERCLEG; ou encore BRASGAUCHE, BRASDROIT, OEILGAUCHE, OEILDROIT, JAMBEGAUCHE, JAMBEDROITE; ect...).

3) L'habileté à organiser des procédures avec d'autres procédures est l'habileté la plus souvent associée à la programmation structurée en Logo. Plusieurs auteurs définissent en effet, implicitement ou explicitement, la progammation structurée comme une habileté à créer des programmes où procédures et sous-procédures sont logiquement et économiquement emboitées. Certes des niveaux élevés de sophistication peuvent être atteints dans ce domaine, mais nonobstant ces niveaux, l'habileté à organiser des procédures réfère toujours à la capacité d'utiliser des touts (des procédures) comme des éléments et à les organiser entre eux pour produire un nouvel "output". Un jeune programmeur qui crée FLEUR en utilisant à répétition son CARRE , un programme OCTO qui appelle OCTOG puis OCTOD, ou encore une TETE, un CORPS, des BRAS et des JAMBES réunis sous BONHOMME sont autant exemples différents d'organisation procédurale. L'organisation procédurale est en quelque sorte une répétition, à un deuxième niveau, de la définition de procédure, à la différence près que les entités alors sélectionnées et organisées ne sont plus uniquement des actions simples (primitives), mais aussi des séries d'actions regroupées (procédures).

4) L'habileté à exporter des procédures réfère pour sa part à l'utilisation répétée et variée d'une même procédure dans plus d'un projet. La procédure ainsi exportée peut avoir été définie isolément ou dans le cadre d'un projet particulier. Les formes géométriques, par exemple, sont à l'occasion créées isolément, puis ultérieurement elles sont réutilisées dans plus d'un projet (FLEUR, POISSON, BIKE, ect...). En d'autres occasions, une procédure créée pour un projet bien spécifique est empruntée pour un autre projet (un SOLEIL, un OISEAU , un effet FIASH). L'exportation d'une procédure n'est pas toujours faisable (souvent à cause de la présence de commandes spécifiques de déplacement) mais cela n'empêche pas pour autant les enfants d'exercer la dite habileté : à défaut de la procédure elle-même, ils transporteront dans un

premier temps l'idée et le mode de construction (emprunt d'une formule de répète ou copie d'une série de commandes présentes dans la procédure convoitée)!

2-Modèle du développement des habiletés prérequises à la programmation structurée

Les habiletés précédemment décrites traduisent tout compte fait cinq propriétés fondemendales de la procédure Logo : la procédure est une entité *définissable, modifiable, transformable, organisable et exportable*. Ainsi un premier palier d'équilibre en programmation structurée est atteint lorsque le programmeur peut voir la procédure comme la somme de ces propriétés. Il n'est certes pas encore expert pour autant dans la gestion de toutes ces caractéristiques, mais il connait, par expérience directe et construction progressive, la polyvalence de la procédure. Le présent modèle tente de décrire les étapes suivies par nos enfants pour apprivoiser chacune de ces caractéristiques, et pour les intégrer progressivement. A l'instar de d'autres modèles développementaux, il respecte la double dimension du passage du concret à l'abstrait (de l'action au concept) et du simple au complexe (de une à plusieurs caractéristiques). La compétence de l'enfant est d'abord expérientielle et distincte pour devenir progressivement notionnelle et intégrée.

Dans une première phase, les enfants s'exercent à définir des procédures, puis tantôt ils les modifient, tantôt ils travaillent soit à les transformer, soit à les organiser. A d'autres occasions, ils apprivoisent l'idée d'exporter des procédures. Règle générale, ces différentes actions sont, à ce niveau, exercées séparément (dans le cadre de projets différents) ou proche en proche (sans anticipation préalable et non nécessairement répétées dans le projet suivant).

A une seconde étape, certaines habiletés sont délibérément pairées pour la réalisation ponctuelle d'un projet. Les trois regroupements les plus fréquemment observés sont : a) définir et transformer; b) définir et organiser; c) exporter (au lieu de définir) et organiser. A noter que l'habileté à modifier une procédure est ici percue comme un outil applicable à toute procédure simple (sans sous-procédure). L'action procédurale est donc, à ce niveau, plus complexe, plus intégrée (référant à plus d'une propriétés à la fois) qu'au niveau précédent. Un exercice répété de ces combinaisons simples permettra aux enfants de se représenter la procédure comme ayant plus d'une fonctions.

A la troisième phase, la procédure est concue comme *potentiellement* modifiable, transformable, organisable ou exportable. Les planifications et les actions du programmeur traduisent le caractère polyvalent désormais attribué à la procédure : F., 11 ans, annonce qu'il va faire un projet lettres "Je vais rapetisser mon CERCLE pour faire mon o, et je vais l'utiliser pour b, d, p, q en leur rajoutant une queue; puis je vais prendre juste une partie du o (DEMI.CERCLE) pour faire c,m,n .....ect. Avec toutes mes lettres je vais me faire un A.B.C. et avec les lettres de mon A.B.C. je vais écrire le nom de mes amis." Et F. mit quatre semaines, à raison d'une heure/semaine, pour réaliser son projet : une super procédure A.B.C. contenant vingt six procédures, souvent parentes entre elles, et quatre autres super-procédures reproduisant les noms d'amis. Cette vision multifonctionnelle de la procédure ne règle pas automatiquement, pour autant, certains problèmes concrets de gestion procédurale (problèmes d'interfaces entre autres), ni plus qu'elle assure d'une maitrise de tous les instants de la logique de la pensén procédurale. Mais à notre avis, les expériences et connaissances acquises constituent la base des compétences en programmation structurée et elles outillent bien le sujet pour aborder la prochaine phase développementale, celle de l'apprentissages des mécanismes et lois de gestions des propriétés de la procédure.

Ainsi le projet d'animaux en formes géométriques de R. illustre bien le chemin encore à faire, mais aussi les compétences déjà acquises pour faire face aux futurs apprentissages. Pour définir ses différents types d'animaux, R. exporte, transforme, organise sans aucun problème des formes géométriques déja définies dans un projet antérieur de banque de formes; mais voilà que R. expérimente toute une série de difficultés dans l'organisation et l'exportation de sous-procédures (oeil, nez ...) définies cette fois au fur et à mesure de ses besoins : les nouvelles procédures incluent souvent des déplacements et s'avèrent plus difficilement exportables, organisables; la position de la tortue diffère selon que c'est l'oeil du poisson ou celui de la chenille qui est à tracer et cela entraine frustration ou modification des plans de travail procédural; ect... Les obstacles rencontrés n'empêche toutefois pas R. de réunir ses trois animaux complétés dans une nouvelle super-procédure. Il y a donc encore des inélégances, des solutions parfois élémentaires, mais la pluralité fonctionnelle de la procédure est désormais connue, expérimentée et appréciée par le programmeur; l'attention et les énergies peuvent désormais être portées sur l'analyse des conditions nécessaires pour que les procédures définies soient en tous temps et toutes circonstances *modifiables, transformables, organisables et exportables.*

La description du présent modèle élaboré à partir d'observations d'enfants de 9-11 ans travaillant à long terme dans un environement Logo visait deux objectifs principaux. Un premier objectif était de souligner le développement effectif des compétences procédurales même si le niveau de performance atteint, pour les âges ici étudiés, n'est pas toujours celui d'un expert. Un second objectif était de dégager certaines idées de travail pour quiconque veut favoriser le développement et l'apprentissage de la pensée procédurale en programmation structurée. Le présent modèle suggère quelques lignes de force à inclure dans un plan d'intervention Logo. La présentation de petites mises en situation où les enfants Logo sont appelés à définir des procédures pour tantôt pouvoir les modifier, tantôt les transformer, tantôt les organiser ou les exporter permet l'identification et l'expérimentation de différentes fonctions procédurales, et ce dans le cadre de situations simples et stimulantes. De même, la présentation subséquente de mises en situations demandant de combiner deux ou trois de ces actions habilite déjà le jeune programmeur à anticiper le caractère multifonctionnel de la procédure et à confronter certains problèmes simples de gestion procédurale. Ainsi outillé, l'enfant peut par la suite faire appel à l'ensemble des propriétés procédurales pour la réalisation de projets, libres ou suggérés, plus variés et plus complexes. Compte tenu du niveau de compétence maintenant atteint, l'apprenti-programmeur pourra désormais spontanément s'attaquer à différents problèmes de gestion procédurale (étant donné les différentes parties d'un projet quel est l'ordre préférentiel des étapes de travail; comment doit-on procéder si l'on veut définir des procédures utilisables dans plus d'un projets; ect..).

BLOUIN, P., LEMOYNE, G. Does the logo environment allows young children to construct programming and mathematical knowledge. Proceedings of the third international conference, Logo Math Education. 1987, pp. 139-147.

KURLAND,M., CLEMENT, C., MAWBY, R., PEA, R., Mappings the cognitive demands of learning to program. In R. Pea, M. Sheingold (eds). Mirors of Minds. Alex Publishing, 1987.

HILLEL, J. Mathematical and programming concepts acquired by children aged 8-9 in a restricted logo environment . Rapport de recherche, Université Concordia, 1984.

HILLEL, J., SAMURCAY, R. , Analysis of a Logo environment for learning the concept of procedures with variables. Rapport de recherche, Univeristé Concordia, 1985.

NOSS, R., Creating a mathematical environment through programming: a study of young children learning Logo. Doctoral thesis. University of London 1985.

THE NAIVE CONCEPT OF SETS IN ELEMENTARY TEACHERS

Liora Linchevski and Shlomo Vinner
Israel Science Teaching Center
Hebrew University Jerusalem

Abstract

Four aspects of the concept of set were examined in 308 elementary teachers and student teachers by means of a questionnaire based on some interviews. The aspects were the following: 1. The set as an arbitrary collection of objects. 2. The singleton as a set. 3. The set as an element of another set. 4. The order of elements in a set and the problem of repeating elements. It was found that the naive concept of sets in these teachers differs from the mathematical concept. The majority of these teachers believes that the elements of a given set should have a common property, that a set cannot be an element of another set and that either repeating elements or the order of elements in a set do count. About a half of them believes that a singleton is not a set.

The naive concept of set seems to us both interesting and important. Everybody who teaches the concept of set at a higher level of Mathematics, whether this is a high school level or a college level, should be aware of the common views about the concept Since the word "set" appears very often in everyday language, it is only natural to assume that almost everybody will have definite views about it which are different from the mathematicians' views.

In this study we chose several aspects of the mathematical concept of sets and examined whether elementary teachers are aware of them and if not what are their conceptions. The reasons we chose elementary teachers were the following:

1. We believe that it is important to know about the mathematical concepts of elementary teachers whether or not these particular concepts are taught directly in school.
2. The naive concepts of elementary teachers are probably quite close to the naive concept of most educated people with limited background in Mathematics. Thus it is possible to assume that junior high, senior high students or even college students, when starting to study about sets, have similar concepts. This assumption, however, needs experimental verification.

Some of the aspects we chose were raised by Freudental (1969a, 1969b, 1970, pp.339-344) and Vaughan (1970). Both of them pointed out that the didactical approach to the concept of set, as presented in many textbooks, is sometimes inconsistent with the mathematical concept. This can be a cause for misconseptions.

Our research questions are the following:

1. Do elementary teachers think that all members of a given set must have a common property? In other words, under what conditions is a collection of objects considered as a set by elementary teachers (the issue was raised in Freudental, 1969b).
2. Is a singleton (a set containing only one element) considered as a set by elementary teachers?
3. Do teachers understand that one set can be an element of another set, and also, when drawing the diagram of the union of two sets, are they aware of the difference between the diagram representing a new set whose members are the two given sets and the diagram which really represents the union?
4. What are the teachers' criteria to determine whether two sets are equal and how are these criteria related to the mathematical criterion? Note that the mathematical definition for set equality is the following: $A=B$ in case for every element x, $x \in A$ if and only if $x \in B$. Thus, repeating elements in lists, tables or diagrams describing sets should be considered as one element and also the order of the elements in such list is not important, namely, lists with the same elements but with different order describe the same set.

Method

Sample

Our sample consisted of 237 elementary teachers (all of them teach mathematics to their students) and 72 student teachers (who were preparing themselves to teach mathematics among other subjects) in Jerusalem. In the 237 teachers we distinguished between two subgroups. The first one included 54 Mathematics coordinators. These are elementary teachers who are interested in teaching Mathematics and in addition to that also underwent some in-service training, thus, their mathematical background is better, to certain extent, than the other 183 teachers who consisted the second group. In the result section we will refer to this second group as the teacher group.

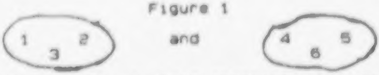
Questionnaire

In order to form our questionnaire we interviewed 21 teachers. We posed to them several questions and recorded their reactions. As a result of this interaction we modified the interview questions and came up with the following questionnaire.

Questionnaire

1. Which of the following collections is a set? Explain your answer!
   - (a) 1, 3, 7, 9, 0, 12
   - (b) a book, 1, 3, a table, 7, 9
   - (c) a table spoon, a tea spoon, a fork, a knife
   - (d) 7
   - (e) all the children under age 10 who flew to the moon
   - (f) (7), (5), 7, 5
   - (g) a triangle, a square, a circle, a box
2. Give five examples of sets which you would choose in order to present to your students the concept of set.
3. A teacher asked her students to give an example of a set. One of the students wrote: My set has three elements: (a) 5, (b) 1.5, (c) the set of all the even integers between 2 and 100. Is this answer correct? explain!
4. Given the sets

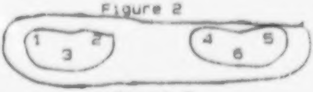
Figure 1



Which form (if any) seems to you more appropriate to represent the union of these sets?

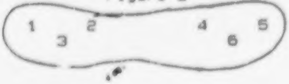
(a)

Figure 2



(b)

Figure 3



Explain!

5. Given the set (1,3,[illegible],7,9). Which of the following sets are equal to it?
   - (a) (5,3,7,9,1)
   - (b) All the odd integers between 0 and 10.
   - (c) (1,5,3,5,7,9)
   - (d) ($\frac{1}{1}$; $\frac{18}{2}$; 3.5 x 2; $\frac{15}{5}$; 3)
   - (e) (9,5,3,0,7,1,5)
   - (f) (9,5,3,1,0)

   Explain!

The reader can easily find which item of which question in the questionnaire is related to our research questions.

Results

1. The set as an arbitrary collection

In mathematics any collection of objects (arbitrary or not, unless you are in a highly sophisticated situation) is a set. However, 97% of the student teachers, 89% of the teachers and 60% of the Math. coordinators did not consider as sets collections whose elements did not have something in common (note that a "teacher" in this section is a teacher who is not a Math. coordinator). A collection of elements is regarded as a set only if these elements have a common property. Many respondents did not accept 1(b) as a set explaining it, for instance, by: 1. *No common property.* 2. *I can't think of a name describing the entire collection.* 3. *There are least two sets here, numbers and objects.*

The last arguments is especially interesting since it implies that a union of two sets is not necessarily a set, a claim which contradicts one of the fundamental principles of Set Theory. Among the arguments not accepting 1(g) as a set we found: 1. *A box is 3-dimensional contrary to the other figures.* 2. *One of the elements does not have a common property.*

On the other hand there were some respondents who accepted 1(b) or 1(g) as sets by saying: 1. *I think that any set of elements can be defined as a set even if they do not have salient common property.* 2. *It is an arbitrary set.* Some respondents who accepted 1(a) as a set said: 1. *Probably this is a union of even numbers and odd numbers* (here, the belief that a union of sets is also a set is expressed contrary to a previous case above). 2. *This is a set of numbers. There is a common property* (although the answer is mathematically correct the explanation shows that the criterion for a collection to be a set is the common property).

It is worthwhile to mention that 17% of the Math. coordinators rejected some items in Question 1 as sets because the parentheses of the set notation were missing. For instance, 1(a) was not considered as a set because it was not written as (1,3,7,9,0,12). Formally, they are right, but formal notation was not our concern in this questionnaire. For the decisive majority of our sample this was not a problem at all, while 23% of the Math. coordinators, 8% of the teachers and 3% of the student teachers admitted that a set can be an arbitrary collection of objects, in the "construction task" (Question 2) almost everybody mentioned sets with common property. This shows how weak is the idea of arbitrary sets (if it exists at all) in these teachers' thought. It is natural to expect examples with common properties when you ask for one, two or three examples, but when you ask for five examples and the respondents are aware of the concept of arbitrary sets and its importance to students they should give at least one arbitrary set. The typical answers were: 1. *Shirts* 2. *The students of the first grade* 3. The girls in the class 4. *{1,2,3,4,5,....}*. It was interesting to examine the answers to Question 3. It was not accepted as a set by all the teachers, all the student teachers and 95% of the Math. coordinators.

In this item there were two potential arguments for the rejection. The first one was the common property and the second one was the claim that a set cannot be an element of another set. It turned out that in this case the second argument was dominant (56% of the teachers, 70% of the students teachers and 76% of the Math. coordinators). The above information is presented in Tables 1 and 2 with some additional information.

Table 1: Distribution of respondents to Question 1

	Teachers (N = 183)	Student Teachers (N = 72)	Math. Coordinators (N = 54)
The elements of a set should have a common property	89%	97%	60%
A collection of arbitrary elements can be a set	9%	3%	23%
The elements should be given in parentheses otherwise they do not for a set	2%	0%	17%

Table 2: Distribution of respondents to Question 3

	Teachers (N = 183)	Student Teachers (N = 72)	Math. Coordinators (N = 54)
It is not a set because there is no common property	44%	30%	19%
It is not a set because one set cannot be an element of another set	56%	70%	76%
Other	0%	0%	5%

2. The singleton as a set

This aspect of sets was examined by item 1(d). The results are shown in Table 3.

Table 3 - Distribution of respondents to Question 1(d)

	Teachers (N = 183)	Student Teachers (N = 72)	Math. Coordinators (N = 54)
A single element cannot form a set	48%	55%	8%
A single element can form a set	52%	45%	92%

Typical explanations to 1(d) were:

1. *No, a set is more than one element.*
2. *Yes, it is a set with one element.*
3. *Yes, a set with 7 elements.*

Note that in answer 3 the inability to accept a set with only one element led the respondent to the concrete interpretation of the number 7: it became a set of seven elements. Unfortunately, we could not locate the respondent to ask her or him whether the number 1 can form a set.

3. The set as an element of another set and the representation problem of a union of two sets

These aspects were examined by Questions 3 and 4. Table 2 already indicated that at least in the context of Question 3, the majority does not accept a set as an element in another set. It cannot be claimed that the figures in Table 2 really express the percentages of those who believe that a set cannot be an element in another set. This is because of the fact that the respondents had 2 options to answer the question. Many of them chose the argument of common property. We do not know what percentage of them, if asked explicitly about this issue, would have accepted or rejected the idea of one set as an element of another set. Thus, we believe that the percentages of those who rejected the idea of a set as an element of another set are higher than those indicated in Table 2.

As to the representation of the union of two sets (Question 4), more than a half prefered Figure 2 to Figure 3 (61% of the teachers, 47% of the student teachers and 50% of the coordinators). According to the mathematical convention, Figure 2 represents a set whose elements are the two original sets. We are sure that this was not the intention of the respondents (most of them do not accept the idea of one set being an element of another set). However, they do not notice that a circle around a list of elements makes it a set according to the common convention and therefore, at the context of Figure 2, they got a set whose elements are the two given sets. The reason why so many respondents prefer Fig.2 to Fig.3 might be that in Fig.2 there is an indication how the union set was constructed from its components.

4. Equality of sets, the problem of repeating elements and order

Question 5 had the potential to examine three aspects of the concept of Set. The first one can be described at the intention-extention aspect. A set can be described in various ways, each of them relates to a different property of the elements. When comparing the sets defined like that, should we pay attention to the properties and thus stipulating that we deal with different sets or ignore the properties and pay attention only to the elements and thus deciding that the sets are equal? In other words, when comparing between sets, should we consider the intention or the extention? For instance, the set of all even primes and the set of all the whole numbers less than 3 and greater then 1 have the same extention but different intention. In Mathematics, only the extension is considered when determining equalities of sets. Thus, $(\frac{8}{8} ; \frac{18}{2} ; 3.5 \times 2 ; \frac{15}{3} ; 3) = (1, 9, 7, 5, 3)$.

Several respondents considered the last two sets as different sets. One can claim that this was done on the basis of superficial impression. They simply did not bother to carry out the computations at the left side. This might be true in some of the cases. In other cases the written explanations showed that the distinction between the two sets was made because of conceptual considerations.

Table 4: Distribution of answers to question 5. The principles used in order to determine the equality of sets

	Teachers (N = 183)	Student Teachers (N = 72)	Math. Coordinators (N = 54)
The mathematical definition	18%	15%	56%
The order and the intention do not matter but repeating elements make a difference	21%	34%	13%
The order does not matter but the intention makes a difference	5%	5%	6%
The order does not matter but intention and repeating elements make a difference	21%	16%	11%
Order repeating and intention each of them makes a difference	32%	28%	6%
Other	3%	2%	8%

The other two aspects which were involved in Question 5 were the repeating elements and the order. The respondents answering Question 5 could fail in each on of the above aspects if they deviated from the mathematical criterion for equality of sets. This is shown in Table 4.

Discussion

Our results show several conflicts between concept images and concept definitions (Vinner, 1983) in the case of sets. Our sample consisted of elementary teachers but every population with the same mathematical background will probably have similar views. It is interesting to compare between the subgroups of our samples (teachers, Math. coordinators and student teachers). There are some items were the Math. coordinators did better and even much better than the rest of the sample (see Tables 1,3 and 4). In Question 4 there was not noticeable difference between the teachers and the Math. coordinators. On the other hand in Table 2 the rejection of a set as an element in another set seems higher in the Math. coordinators. This impression, however, might be wrong. It might be the result of the fact that Math. coordinators do not deny a collection from being a set on the ground of not having a common property. For certain percentage of the rest of the sample this is still a good reason.

The fact that the Math. coordinators showed better conceptual understanding is enough ground for hope that teaching can overcome some students' primary views. But teaching can be much more efficient if it relates to the primary views which were described here and does not ignore them as it does in many cases. Our recommendation is that studies like this one should be presented to student teachers in an appropriate way when they are taught about sets. This will help them to overcome the misconceptions they already have or those that might develop if certain steps of caution are not taken ahead of time.

References

- Freudental, H., (1969a). Braces and Venn Diagrams. **Educational Studies in Mathematics**,1. 408-414.
- Freudental, H., (1969b). A Teachers Course Colloquium on Sets and Logic.**Educational Studies in Mathematics**,2, 32-58.
- Freudental, H., (1970). **Mathematics as an Educational Task**. Reidel Publishing Company.
- Vaughan, H., (1970). What sets are not? **Arithmetic Teacher**, 17, 55-60.
- Vinner, S., (1983). Concept Definition, Concept Image and the Notion of Function. **International Journal for Mathematical Education in Science and Technology**, 14(3), 293-305.

CONCRETE INTRODUCTION TO PROGRAMMING LANGUAGES AND OBSERVATION OF PIAGETIAN STAGES.

- CLINICAL INTERVIEWS -

F. Lowenthal
Université de l'Etat à Mons.

We introduced, in a concrete fashion, a simplified programming language to very young children. The device we used can be used to train young children (6-year olds) in a very specific task, but also to observe trained and untrained children during clinical interviews. The training does not seem to have any immediate influence on purely school performances ; but the clinical interviews show that the trained children have acquired skills which are not natural for children of that age : the ability to use and combine two inputs to produce an output, some notions about recursion and programming. We wonder whether there is any transfer to other domains such as : Piagetian conservation tasks, the use of names instead of a long and complete description, the study of real programming with a language such as LOGO.

PIAGET has shown (1936, 1955) the importance of concrete manipulations at an early stage. He used such manipulations to observe how children acquire concepts such as conservation of liquids, ... FLAVELL (1977) wrote that one of the major differences between the pre-operational and the concrete operation stages is that the younger child is centered on one relevant element of the phenomenon he observes, while the other child is "decentered" and can consider several relevant elements simultaneously in order to compare them and draw logical conclusions. BRUNER (1966) described an experiment where concrete manipulations of "logically organized" objects were used (beakers were placed at different places on a board depending on their height and thickness). He showed that such manipulations made cognitivists' observations easier; but also that "what is needed for the child ... is organizing experiences into a form that allows more complex language to be used as a tool not only for describing it but transforming it".

PAPERT (1984) insisted on the importance of computer languages, and more specifically on the manipulations of representations of objects via LOGO, which is procedural and recursive. Nevertheless all these computer languages, LOGO included, seem too abstract for young children : a special vocabulary must be used, words must be written and read, and a keyboard must be used to communicate with the computer. Even if we assume that these minor details can easily be settled, we still do not know wether young children are able to use a language such as LOGO : do they really conceive what a procedure is ? are they able to replace by a name a list of actions, and then to combine such names instead of combining basic actions ? do they have any understanding of the concept "recursion" ? Should the answer to one or more of these questions be negative, one might wonder whether these concepts and competences can be taught to young children and then wonder wether such a teaching would be useful.

For all these reasons, we thought that it would be more useful to let young children manipulate concrete representations of objects which suggest a logical structure because these objects are in fact a concrete representation of a formal system sufficient to perform reasonings.

COHORS-FRESENBORG's Dynamical Mazes (1978) can be used in such a way. We used them (1986) with 6-year olds (first graders) and noticed a transfer from the training we gave them to their performances in reading activities. We then tried another material with similar children. This material has been described by SAERENS (1984) who wanted to use it to analyse sentences while we described how to use it as basis for a programming language (LOWENTHAL, 1985).

The device itself consists of a white plastic board furnished with holes. In these holes one can put coloured plastic nails, or pegs. The pegs are defined by two variables : their colour and the shape of the head. There are five colours : yellow, green, red, orange, blue ; the heads can be squares or triangles. We used short sequences of pegs : each received a name represented by a triangular peg ; this definition was placed on the left of the board : our short sequence became thus a procedure. A short sequence of triangular pegs placed in the centre of the board represented a programme : a list of procedures which had to be executed. The end product was then placed on the right and could only contain square pegs (i.e. one had to perform a list, finite or not, of basic actions). We introduced a special directional peg : the yellow triangle which was only used in this context.

The most relevant feature of this device, when used as basis for a programming language, is that it constitutes a procedural language : procedures can be combined and referred to. Another relevant feature of this language is that a procedure can call another one (the name of the other one has been inserted in the definition of the procedure). A procedure can thus call itself provoking an infinite recursion. Finally some kind of turtle-like orientation can be introduced.

We used this setting to ask three kinds of questions. Firstly, we gave the procedures and the programme and we asked the subject to produce a long sequence of square pegs by replacing each triangular peg by its "meaning" (i.e. the subject had to execute the programme). Secondly, we gave the procedures and the end product, and we asked the subject to propose,using **triangular pegs**, a programme which could have been used to produce this end product with these procedures. Thirdly, we gave the programme, the end product and the names of the procedures, and we asked the subject to discover definitions which could have been used for these procedures (i.e. produce a sequence of "things" for each of the given triangular peg). In each case, the child had to solve a problem : he had to produce an output taking simultaneously account of two inputs of different kinds.

The use of such a material as observation and/or teaching device suggests a great number of questions. When comparing children who were trained to use this device with untrained subjects, can one show that the first ones :

a) have better school results, as far as classical school problems are concerned, when they are evaluated by means of classical tests or by the teachers' grades ;
b) have transferred the competence acquired in a pseudo-computer language to reading skills, as mentionned for the Dynamical Mazes, or to vocabulary or other skills involving the semiotic function ;
c) have a better apprehension of spatial concepts ;
d) verbalize more easily and are more able to explain what they did and why ;
e) are more efficient when they start with real LOGO.

One can also wonder whether clinical observations realised while using this material as "testing device", give informations concerning the type of cognitive strategy a child uses and the type of cognitive processes which are involved for him (COHORS-FRESENBORG, 1984 ; SCHWANK, 1986).

Finally we will show that such observations can be used to specify which Piagetian stage, (or part of a stage) has been mastered by the subject. Moreover we will show that some higher concepts are teachable to younger children although we do not yet know whether the result of our teaching is limited to the use of this material or can be transferred to other domains.

1. EXPERIMENTAL SETTING.

We worked with 76 6-year olds (first graders). We used a pre-test to split the group in two equivalent subgroups : an experimental group and a control group. The pre-test contained three parts : a) a reduced version of the BD in order to measure the general intelligence of the subject (in VAN WAYENBERG), b) PORTHEUS'mazes in order to measure the subject's capacity to make plans and to foresee, and c) Similitudes, items extracted from the WPPSI (WECHSLER, 1972).

The "experimental" subjects worked during 6, 7 or 8 30-minutes sessions with the pegboard. They worked by groups of two. The "control" subjects also worked by groups of two on typically placebo activities (play games, draw, sing, ...). This activity lasted from January to May.

At the end of this activity, all children were submitted to a post-test containing four parts : a) a reduced version of the BD, b) LAMBLIN's "test de la goutte" (in VAN WAYEMBERG), similar to the REY figure test, but simpler : it measures the level of structuration of the perceptive-activity competence, c) the Reversal (EIFELDT, 1970) which measures the level of spatial organisation and lateralisation and d) a test of mathematical knowledge (CLEEMPOEL-HOTYAT) concerning only the kind of mathematics which should be taught in a first grade. 6 months later, in January again, we interviewed the 11 children who got the best scores at the post-test, in each group. The procedure used for these clinical interviews will be presented later in this paper.

2. STATISTICAL RESULTS.

As far as typical and classical school activities, problems, mathematical activities are concerned, our testing shows NO significant difference between the experimental and the control group. the teachers' evaluations of reading abilities were also taken into consideration (but this is not a standardized test) : these evaluations do not show any significant difference between both groups. Neither did we observe a difference in the test which was used to measure the subjects' apprehension of spatial concepts.

3. CLINICAL INTERVIEWS.

We prepared for these interviews a video tape showing an adult hand solving 6 exercises on the pegboard :

a) production of the end product ;
b) discovery of the programme used ;
c) definition of the needed procedures (the solution of this exercise, as presented on the videotape contained a mistake) ;
d) use of procedures calling other procedures ;
e) use of a procedure containing its own name (and thus calling itself) ;
f) use of the directional triangle.

The material was shortly presented to each subject when he started to view the tape. The interviewer showed him then what the adult had done stopping at each step (and he thus subdivided an exercise in as many parts as requested by the child) ; he asked the subject to tell what he had seen and to explain what had been done and why it had been done. Some children were also asked to predict what would happen next and all the children who seemed unable to understand what was going on got hints from the experimenter.

In order to analyse these clinical interviews we looked at the following elements :

a) what type of explanation does the subject use to produce or explain a result : we consider that a subject produced a high level explanation if he took into account 2 information sources constantly, or 1 source for some problems but 2 sources for most of them ; otherwise we consider that he produced a low level explanation;
b) is the subject able to use labels to represent a collection of objects (e.g. a chain of squares) and to manipulate these labels instead

of the objects they represent, in order to perform a reasoning ;

c) is the subject able to understand and explain that a procedure can contain a label (a triangle) "calling" another procedure ;

d) is the subject able to understand and explain that a procedure might contain its own name and thus provoke an "infinite recursion" ;

e) is the subject able to discover the mistake made by the adult and to react by correcting it rather than by modifying his own solution ;

f) is the subject able to understand and explain the meaning of the directional triangle ?

- In the experimental group, 7 subjects (out of 11) gave high level explanations, the same subjects used labels in a useful fashion, the other 4 subjects gave some kind of explanation ; in the control group 3 subjects (out of 11) used high level explanations and 2 subjects gave explanations taking none of the available information into account, the only control subject who used labels correctly, also gave high level explanations.
- In the experimental group, 5 subjects clearly understood that a procedure can contain a label, and thus call another one and 3 of them more or less understood the process associated with the "infinite recursion", but 3 of the 5 subjects mentionned above had never seen similar problems during the training period ; in the control group none of the subjects understood either the "call" or the "infinite recursion".
- Most children either did not detect the mistake or modified their correct proposal to stick to the wrong adult solution.
- In both groups, 3 subjects understood the meaning of the directional triangle, only 2 of these 6 children had seen similar problems during the training period.

4. DISCUSSION AND PROPOSALS FOR FURTHER RESEARCH.

A. Reflections concerning the subjects of the experimental group.

- These subjects have been trained to perform a very special kind of task which has no relation with what is usually done at school. More than 6 months after the training ended, these subjects perform well when they have to discuss and explain this task to an adult. It is thus obvious that these learners have assimilated certain notions and/or strategies. It is thus important to try to specify which notions or strategies they acquired and to figure out whether they transferred these to other domains.

- These subjects learned to use labels and to manipulate them instead of the chain of objects they represent : they were thus able at the age of 7, to manipulate symbols *in a specific setting*. A further experiment whill show whether they do this in other domains, and more specifically whether they use less, as much or more periphrases and metaphors in their usual language, than the subjects of the control group.
- These experimental subjects also seem to be able to use, *in a special setting* and at the age of 7, two different sources of information simultaneously, and to combine them in order to explain a fact. A further experiment will show wether these subjects are better than their control counterparts when confronted to typically piagetian conservation tasks requiring the *ability* to combine two informations (e.g. width and height of a glass).
- Finally some of these subjects appear able to understand that a procedure can contain basic instructions *and* instructions "calling" another procedure. This is also the case for experimental subjects who never saw similar exercises before. A research with LOGO on actual computers will show whether they are better at programming tasks with LOGO or simply better in the direct mode, or not better at all.

B. Reflections concerning the subjects of the control group.

- The results obtained by these subjects show that, at least in the setting we used and more probably in general, certain activities are not natural before the age of 8 : e.g. use simultaneously and combine two informations to explain or produce an output ; use and manipulate labels instead of the objects themselves ; use in a sequence of instructions the instruction : "perform the instruction whose label is xxx". A further experiment will show at which age these activities become natural for non trained subjects. It has been shown (LOWENTHAL and EISENBERG, 1984) that the use of recursive reasoning is not always natural, although it is necessary, in 18 year olds students starting a University course in mathematics.

C. Reflections concerning all the subjects.

- Our observations also confirm that children aged 7, experimental and control, are easily impressed by adults' solutions and do not react positively to adults' mistakes.

- A less expected observation is that the meaning of the directional triangle has not been discovered by a majority of subjects, although most of them agree to say that "the squares (end product) do not look the same" (i.e. are no longer in one line).

We are already conducting the new experiments we mentioned and we believe that the material we described might possibly be used in the future to test children's abilities in totally different problem solving situations. This device will probably also help us to determine which piagetian stage has been reached by the subject.

5. BIBLIOGRAPHY.

- **BRUNER J.S.** : 1966, On multiple ordering, in Studies in cognitive growth, Eds. BRUNER J.S. et al., John Wiley, New York, 155-167.
- **CLEEMPOEL L. et HOTYAT F.** : Tests d'arithmétique pour la première et la deuxième années d'études, Institut Supérieur de Pédagogie du Hainaut, Morlanwelz.
- **COHORS-FRESENBORG E.** : 1978, Learning problem solving by developing automatic networks, R.P.A., 46/47, 93-99.
- **COHORS-FRESENBORG E. and KAUNE C.** : 1984, Sequential versus conceptual -two modes in algorithmic thinking, in Proceedings of PME 8, 216-267.
- **FLAVELL J.** : 1977, Cognitive development, Prentice-Hall, Englewood Cliffs.
- **LOWENTHAL F.** : 1985, Pegboard as basis for programmation - in 5- and 6- year olds -, in Proceedings of PME 9, 47-52.
- **LOWENTHAL F.** : 1986, NVCDs are structuring elements, in Proceedings of PME 10, 363-368.
- **LOWENTHAL F. and EISENBERG T.** : 1984, About recursive thinking - on the modelling-down heuristic -, in Proceedings of PME8, 283-288.
- **PAPERT S.** : 1984, Mindstorms, Basic Book, New York.
- **PIAGET J.** : 1936, La naissance de l'intelligence chez l'enfant, Delachaux et Niestlé, Neuchâtel et Paris.
- **PIAGET J. and INHELDER B.** : 1955, De la logique de l'enfant à la logique de l'adolescent, P.U.F., Paris.
- **SAERENS J.** : 1984, Semantic Components and Syntactic Sequences Introduced by Means of Multicolored plastic pegs, Communication and Cognition, 17, 4, 397-403.
- **SCHWANK I.** : 1986, Cognitive structures of algorithmic thinking, in Proceedings of PME 10, 195-200.
- **VAN WAYENBERG M.A.** : Test 6 ans (EP1) pour l'entrée dans le cycle primaire, Clerebaut F., Bruxelles.
- **WECHSLER D.** : 1972, Labyrinthes et Similitudes, W.P.P.S.I., Editions du Centre de Psychologie Appliquée, Paris.

This research has been partially realised thanks to an [illegible] grant.

COGNITIVE and METACOGNITIVE SHIFTS

John H. Mason

P. Joy Davis

file:PME12

ABSTRACT

A basic mathematical question is to ask, overtly, 'What is the same about these apparently disparate and yet strikingly similar situations?', and to try to bring this to articulation. PME XI in Montreal provided us with numerous apparently disparate experiences, in the form of the many presentations. Yet we were struck by a common thread running through most of the sessions we went to, and this paper is an attempt to articulate that sameness.

The sameness has to do with shifts in perception and attention. One example is the shift indicated above, in moving from a sense of sameness, to an articulation of that sameness. The sense of sameness is akin to breathing air - a natural activity, a state of immersion in experience. Becoming aware of sameness as a sameness, and trying to bring that to articulation, is akin to becoming aware of the fact of breathing, and trying to describe what breathing is like. Our intention is to go further, and to begin an analysis of the mechanics and function of shifts, akin to studying the mechanics and function of breathing.

This paper must necessarily be brief, and hence laconic. A fuller analysis, with more examples, with more detailed links made between examples and mechanism, with an exegesis of the kind of theory which we are developing, and with a justification for our epistemological approach and our method of study, must wait for another occasion. Elements can already be found in Mason & Davis 1988 and Mason 1986. There are four sections:

1 The scope and range of shifts in mathematics education:
  Examples of some of the fragments of disparate experiences which the idea of shifts embraces.
2 The fundamental importance of shifts in the psychology of learning mathematics.
3 Significant factors in the bringing about of shifts:
  First steps towards a descriptive vocabulary to enhance noticing.
4 The structure of attention:
  First steps towards a mechanism of shifts.

1 THE SCOPE AND RANGE OF SHIFTS IN MATHEMATICS EDUCATION:

Examples of some of the fragments of disparate experiences which the idea of shifts embraces.

By a *shift*, we mean a shift of attention, often sudden, but sometimes gradual, in which one becomes aware that what used to be attended to was only part of a larger whole, which is at once, more complex, and more simple. Frequently, shifts studied in mathematics education are from object to process, and from process to process-as-object. For example:

- a shift from attention to number as a sound uttered during the 'act of counting', to attention to the act of counting, and then to number as independent of counting;
- a shift from having to mentally calculate when converting, say, temperature given in degrees centigrade to degrees fahrenheit, to simply knowing (perhaps approximately), in both systems;
- a shift from root 2 as a number approximately equal to 1.414..., to root 2 as a number known only by its property that it is positive, and that its square is 2.
- a shift from seeing an infinite sequence as an unending process in time, to seeing it also as a completed act;
- a shift from experiencing emotions while engaged in mathematical thinking, to being aware of affective factors in mathematical thinking during the thinking;
- a shift from being immersed in being stuck while working on a problem, to being aware of being stuck, and hence freed to be able to do something about it;
- a shift from seeing a mathematical problem as being hard, interesting, important, ..., to seeing 'hard', 'interesting', 'important', etc as descriptive of the relationship between a person, a problem, and the circumstances;

A quotation attributed to Kant, sums up beautifully the essence, the ubiquity, and by extension, the importance of shifts: *The succession of our perceptions does not add up to a perception of that succession.*

2 THE FUNDAMENTAL IMPORTANCE OF SHIFTS IN THE PSYCHOLOGY OF LEARNING MATHEMATICS

The examples of shifts given in section 1 are intended to be immediately recognisable to mathematics educators. They illustrate some of the aims and activities reported by many researchers at PME XI, some of whom were concerned with how pupils learn specific mathematical ideas, concepts, and techniques; some of whom were concerned with how teachers might intervene with pupils to facilitate learning; and some of whom were concerned with helping teachers to become aware of their own thinking processes and thus in turn to help their own pupils.

We suggest that to make contact with a mathematical idea, to learn a concept, to master a technique, and to develop an awareness, all require a shift of perception in the pupil, indeed, often several shifts. For example, in the well studied domain of algebra, which is a watershed for most people, there are at least five fundamental and essential shifts required:

from an expression seen as a complex entity, to being seen both as a rule for calculation and as the result of a calculation;

from attention on the result of counting, to attention on the act of counting, so as to discern the generic aspects of the counting;

from single right/wrong answers to the possibility of a multiplicity of ways of expressing the same pattern;

from the unknown as unknown, to the unknown being merely a manipulable as-yet-unknown (Mary Boole 1909);

from 'seeing' pattern, to pictures supporting that 'seeing', to words describing that 'seeing', to succinct words, to symbols which can conveniently be manipulated.

The charting of common pupil misconceptions can be viewed as a charting of behaviour in the absence of necessary, but sadly, essential shifts of attention. Teachers try to encourage pupils to shift their attention, from focussing solely on getting correct answers, to *how* such answers are obtained, and thence to the processes of thinking mathematically. Teahcers often find themselves encouraging shifts of attitude, which is concomitant with attention, among colleagues and parents, as well as pupils. Educators conducting in-service sessions with teachers are trying to encourage teachers to shift their attention away, for example, from mathematics as fact-learning and towards mathematics as engaging in thinking and as a disciplined form of enquiry. At a second level, they wish to help teachers shift their attention away from the details of specific lesson plans and detailed tips for good lessons, and towards a general approach to teaching.

What we found most striking, in discovering the idea of a *shift of attention* lying behind the wide variety of research, pedagogic and inservice activity displayed at PME XI, was that there is comparatively little information about how such shifts of attention actually come about. People report on their perception of teacher and pupil behaviour, but tend to leave unnoted sufficient details to enable a study of what brought about such attention shifts as do take place.

Before proceeding with our suggestions, it is extremely important to draw attention to an enormous potential danger in the use of the language of shifts. The English language encourages reification of processes, and mathematics often makes progress by making just such a shift. Shifts could become things which 'have to be made to happen'. The very word *shift*, based as it is on a spacial metaphor, suggests that it is something that you can 'do' to someone else. The next stage in the potential degeneration of ideas through excessive articulation, is that teachers might start to try to 'shift pupils', educators to 'shift teachers', and researchers to study all this 'shifting' activity. We believe that the notion of shifts is sufficiently important and powerful to take that risk, but we emphasise that shifts are NOT something you do to someone else. You cannot shift someone else's attention. You may attract it, you may try to focus it, you may even act in a manner which invokes temporary shifts of perspective. But, based on our experience, we are certain that you *cannot* shift someone else's attention, at least in the way in which we are using that term.

What is the use of a theory which denies the possibility of causation in its application? We suggest that through the language of shifts, it becomes possible to notice situations in which shifts, and blocks to shifts, are significant factors, and because of this awareness, alternative action can be taken - for example, in not 'beating your head against a wall', but rather setting up activities that might promote the necessary shifts (for example, the Didactic Situations of Balacheff 1980). By focussing attention away from the teacher as curriculum delivery agent, and towards the teacher as guide and gardener, the vocabulary connected with the theory of shifts can help influence the development of a more productive classroom environment. Notice that we are here talking about a shift of attitude and perspective, connected with a shift in focus, on the part of the teacher. Our theory (when fully elaborated) speaks to its own promulgation.

3 SIGNIFICANT FACTORS IN THE BRINGING ABOUT OF SHIFTS:

First steps towards a descriptive vocabulary to enhance noticing.

Working from experience of ourselves, from observations of and discussions with others, and reflecting on the examples proffered so far, it seems that shifts come about in basically four ways:

in the presence of a person, usually whom we esteem or in whom we have some investment (*Investment* for short);

when present experience is suddenly seen as an example or particular case (*Examplehood* for short);

when a word, expression or image which is richly associated with past experience (often described as meaningful or substantial) provokes a moment of noticing (*Resonance* for short);

when we suddenly, and apparently spontaneously notice something new or freshly (*Grace* for short).

Several of these may be operating at the same time. The reasons for distinguishing and labelling them are that we can elaborate on those aspects of shifts which seem to fit these patterns, and the labels can be used (via the mechanism of resonance) to help notice shifts taking place, thereby permitting specific action to be chosen.

Investment

Try to recall some moment when someone whom you respected or esteemed came for the first time to your room or other familiar place. Often when this happens, it is as if you see the place freshly, perhaps even through the other person's eyes. Sometimes there is a sense of being larger than life, of being more than ordinarily aware. It can also be dysfunctioning in that you find yourself knocking things over or otherwise behaving awkwardly. Teachers being inspected or visited often report this sort of experience.

We suggest that personal investment describes the principal action behind many metacognitive shifts. Such shifts occur when attention is effectively split by seeing the world as though through the eyes of another. The investment of esteem literally places part of our attention outside ourselves, and so produces the inner separation which is one form of shift.

Examplehood

Try to recall some situation in which you suddenly realised that what you were attending to was an example or particular case of a general principle. For example,

> counting the number of stairs in a staircase is an example of the 'fence-post-argument';
>
> realising that the experience of suddenly emerging from being stuck on a problem by becoming aware of being stuck, is an example of what we mean by a shift.

The shift to examplehood is remarkably, and peculiarly, hard to speak about, because the act of speaking entails that examplehood has already occurred. Yet there are countless acts that people perform each day, whose examplehood passes unremarked. People say 'Good morning', but don't see this as an example of 'stroking' (Berne 1955); they think about what they will do during the day, but don't see it as 'planning', and so on. We are not suggesting that it would be helpful to see every act, every object, as an example of something more general. However we do observe that in mathematics, many students act as if they have not detected examplehood when it is expected or intended.

Examplehood is an important part of our story, for it describes the way in which disparate experience is integrated into a more substantial, more meaningful net of connections and associations. Along with *making distinctions*, it seems to be a fundamental power of the human brain, and at present represents a 'psychological primitive' (DiSessa 1987) in our theory.

Resonance

In the midst of a conversation, someone uses a word which for you has a technical or emotive importance. Suddenly you both hear what they are saying, and simultaneously, you have an expanded inner sense of the special meaning for you. It often happens that after your return from a holiday in another county, you notice numerous references to that country in travel write-ups and even in the news. A car salesman observed that when you buy a new car, you suddenly become aware of other cars on the road of the same model. These are examples of a process which seems to work rather like the resonance of a musical instrument - if you make a sound in the right place, the instrument reverberates and amplifies the sound. In terms of memory and meaning, a sight, sound or thought can resonate with past experience, making both specific images and abstracted awareness seem to appear in attention. Workers in other disciplines use the language of frames, schemes and scripts to talk about the same sort of experience. The metaphor of resonance does not answer the question of mechanism, but seems a useful way to speak about a whole gamut of experiences, in which something becomes meaningful.

Resonance seems to lie at the heart of many cognitive and metacognitive shifts. The sudden insight, the change in viewpoint, seem to be related to prior experiences which, although not summoned directly, contribute to the shift of attention. This is the 'mechanism' proposed in Mason, Burton and Stacey 1982, for 'learning from mathematical experience' via the use of *emotional snapshots* which are re-vivified fragments of recent significant experience.

Grace

Every so often, in our experience, we suddenly find our attention sharpened, but for no apparent reason. There may in fact be a chain of subtle resonances and associations, but in order to leave room for the possibility of spontaneous shifts of attention, a fourth term seems desirable. We use the word *grace* rather than hazard or chance, because it often seems like a gift, a special moment in which attention is enhanced and 'things seem to fall into place'. Since the act of grace does seem to be haphazard, there is little more that can be said, and certainly it cannot be called upon or planned for!

4 THE STRUCTURE OF ATTENTION:

First steps towards a mechanism of shifts.

Our current understanding of cognitive and meta-cognitive shifts is best stated in terms of splitting and diffusing of attention, from monadic, through dyadic, to triadic form. The transformation of attention has, we propose, qualities analogous to physical change of state, with the role of latent heat being taken partly by stimulation from the environment, and partly by the self, working on automatising and integrating awarenesses (Gattegno 1962, Maturana & Varela 1971). Monadic attention is a state of total immersion and full concentration, of being caught up in the doing and being blissfully unaware. Dyadic attention emerges as awareness of distinction, duality, or identity. Ideas suddenly fall into place, and one becomes aware of the fact of thinking. The focus of attention becomes itself an example or generic instance. This is typical of cognitive shifts connected with mathematical content. Significant metacognitive shifts arise when attention becomes triadic, sometimes through emergence of investment in a significant other, an internal watching bird (Rig Veda c1500BC) or an internal monitor (Schoenfeld 1985, Mason et al 1982), and sometimes through resonance with significant key words or phrases.

The whole of the theory is summed up for us in the memorable epigram of Gattegno (1971): 'Only Awareness is Educable'.

References

Berne, Eric 1964 Games People Play, Penguin, Hammondworth UK

Boole, Mary 1909 Philosophy and Fun of Algebra see also Tahta, Dick 1972 A Boolean Anthology, Association of Teachers of Mathematics Derby

Balacheff, Nicholas 1987 Processus de Preuve et Situations de Validation, Ed Studies in Maths 18 p147-176

diSessa, Andrea 1987 Phenomenology and the Evolution of the Intuition, in Problems of Representation in the Teaching and Learning of Mathematics, ed Claude Janvier, LEA Hillsdale USA

Mason, John 1984 Towards One Possible Discipline of Mathematics Education, in Theory of Maths Edn, Proceedings of TME topic area, ICME 5, ed Georg Steiner, Inst. fur Didactik der Math der U. Bielefeld Occasional paper 54 p 42-55

Mason, John, Burton, Leone, Stacey, Kaye 1982 Thinking Mathematically, Addison Wesley London

Mason, John & Davis, Joy 1987 The Use of Explicitly Introduced Vocabulary in Helping Students to Learn, and Teachers to teach in Mathematics, PME XI vol 3 p 275-281

Maturana, Humberto & Varela, Francisco 1972 Autopoesis and Cognition: The realization of the living, Reidel, Dordrecht

Schoenfeld, Alan 1985 Mathematical Problem Solving, Academic Press London

Rig Veda c1500 BC translation from Bennett, John 1968 Energies, Coombe Springs Press London

LEARNING MATHEMATICS COOPERATIVELY WITH CAI

Zemira Mevarech
Bar-Ilan University, ISRAEL

The purpose of the present study was to investigate the effects of computer assisted cooperative learning on mathematics achievement and learning processes. Participants were 227 pupils in elementary school who studied mathematics with a Computer-Assisted Instruction program called TOAM. Results showed that collaboration at the computer tended to be associated with a higher level of mathematics achievement and more time-on-task than did the individualized CAI program.The theoretical and practical applications of the findings for the psychological aspects of mathematics education will be discussed.

In recent years, researchers and teachers have started to question the widely accepted assumption that Computer-Assisted Instruction (CAI) works best in individualized settings (Johnson, Johnson and Stanne, 1985). Jackson, Fletcher, and Messer (1986), for example, showed that more than 50% of the teachers in England use CAI in pairs or small groups. Jackson and her colleagues indicated that teachers prefer to implement CAI cooperatively not only because of limited sources, but also because of their belief that students benefit more in cooperative than in individualized CAI settings. This assumption raises two important questions: (a) does Cooperative CAI (C-CAI) facilitate learning more than individualized CAI (I-CAI)? and (b) to what extent are learning processes different in one setting than in the other? The present study addressed both questions by focusing on mathematics achievement and mental effort of elementary school students who used CAI cooperatively versus individually.

Underlying cooperative learning models is the fundamental assumption that learning together improves knowledge-acquisition more thar competitive and/or individualized learning (Sharan, 1980; Slavin, 1980). This assumption stemmed from cognitive and social-psychology theories. From cognitive point of view, learning together provides ample opportunities for students to verbalize the material, reorganize it in new schema, and represent it in different ways. According to Webb (1982), these processes facilitate learning. Moreover, research has shown that both high- and low- ability students benefit from cooperative learning (Stallings and Stipek, 1986). The high-ability learners achieve a higher level of understanding via the process of teaching the slow learners; the low-ability learners benefit from the instant help they receive from other children in the small-group. These processes exist also in cooperative mathematics classrooms. When children solve mathematics problems in small groups, they help each other to analyze the problem, identify the "given" and the "wanted", look for appropriate algorithms, and correct computational mistakes.Thus, we hypothesized that students in cooperative CAI settings would perform better than students in individualized CAI settings.

From social-psychology perspective, cooperative learning is presumed to raise motivation and increase mental effort more than individualized or competitive learning (e.g., Slavin, 1980). Stallings and Stipek (1986) argued that "individual competition can enhance the motivation of students who have some possibility of "winning", but research shows that many children, who begin the competition at a disadvantage and who expect to fare poorly, no matter how hard they try, eventually cease trying

(Covington and Berry, 1976; Dweck and Reppuucci, 1983). A group reward structure may relieve motivation problems that many low-ability students have in individual competition situations" (p. 746). If indeed, cooperative learning raises motivation, there is reason to suppose that students in C-CAI settings would invest more mental effort than their counterparts in I-CAI settings.

While most studies of cooperative models focused on settings with no computers (e.g., Sharan, 1980; Slavin, 1980, Stallings and Stipek, 1986), studies in the area of CAI assessed the effects of the system on students who worked individually at the computer (e.g., Kulik, Bangert and Williams, 1983; Mevarech, 1985, Mevarech and Rich, 1985; Mevarech and Ben-Artzi, 1987). Only two studies investigated the effects of CAI in cooperative settings (Johnson et. al., 1985; Mevarech, Stern and Levita, 1987), but they did not examine mathematics learning. The purpose of the present study is, therefore, to compare the effects of C-CAI and I-CAI on mathematics achievement and on mental effort investment.

METHOD

Subjects

Participants were 227 Israeli students in third and fifth grades. Subjects studied in two elementary schools which served economically disadvantaged families as defined by the Israeli Ministry of Education.

CAI Program

The CAI program used in this study is called TOAM, the Hebrew ac nym for Diagnosing and Practicing with Computers.. TOAM program "c ers" all topics of elementary school mathematics including four basic op ations with natural numbers, negative numbers, fractions and d imals; powers; word problems; equations; and geometry. The program is ided into fifteen strands each includes problems of varying difficulty. At every session, problems from all strands are presented on the screen. Students are provided with three attempts to solve a problem correctly. When all three attempts are failed, the correct answer is presented on the screen.

The first ten TOAM sessions are devoted to diagnosing purposes. Using the "testing-tailored" technique, the level of each student is determined independently of his or her age or class level. Then, each student drills and practices according to his or her ability level. The computer makes moment-to-moment decisions regarding the matching of student ability and problem difficulty levels. The criterion level of mastery is approximately 80% correct answers. At the end of a session, students receive summary reports indicating the number of problems provided and the number of problems solved correctly on the first attempt. Teachers and principles receive weekly reports describing performances of all students on every strand and the average performance of the class. In addition, teachers receive information about special problems students are confronted while working on the tasks. (More details about TOAM can be found in Osin, 1981).

Measures

Two measures were used in this study: one focused on mathematics achievement and the other on students' mental effort. Each was administered at the beginning and the end of the study.

TOAM's average scores overall strands were used to assess students' mathematics achievement. The scores are constructed of two-digit numbers. The "tenth" digit presents the "class" level and the "unit" digit presents the "month" level within that class. For example, a student whose score is 54 knows that his or her performance is equivalent to the performance expected by a student at fifth grade on the fourth month. As mentioned earlier, these scores are based only on students' performances regardless of their age or class level. For example, students can be in second grade and perform as expected by students in fifth grade and vice-versa, students can be in fifth grade and perform as expected in second grade. The norms were determined by the Israeli Ministry of Education.

Students' mental effort investment was assessed by a short questionnaire. Following Salomon (1983), students were asked to indicate the extent to which they invested mental effort during the CAI sessions. The Scale ranged from one (very little) to five (very high).

Procedure

At the beginning of the experiment students were randomly divided into an experimental and control groups. Students in the experimental group worked in pairs at the computer. They were asked to discuss the problems presented on the screen, agree on the solution, and then ENTER the answer. In this group, students took turns at the keyboard so that at each session another team-mate typed the answers.

The control group continued to work at the computer as they were used to do. They learned individually -- one student at one computer.

Both the experimental and control groups used the same textbooks, CAI system, and basic teaching methods. The duration of the study was approximately three months.

RESULTS AND DISCUSSION

Results showed that students in the experimental group gained higher mathematics scores than students in the control group. During the time of the study, students who worked individually at the computer gained 2.86 month equivalent grades, whereas students who worked in pairs at the computer gained 3.10 months equivalent grades. Analysis of Covariance (ANCOVA) of mathematics achievement obtained at the end of the study (initial scores served as covariance) indicated marginal significant main effect for the "treatment".

Results also showed that students who used CAI in pairs invested nore mental effort than students who used the program individually. While hanges between post and pre measures of students' mental effort nvestment in C-CAI settings averaged at .37, that of students in I-CAI ettings remained almost the same (.08). ANCOVA of mental effort scores tained at the end of the study (initial scores served as covariance) dicated significant main effect for the "treatment".

Since this work is now in progress, more details will be communicated the PME meeting in July.

These findings support teachers' intuition that C-CAI facilitates learning more than I-CAI. Evidently, students collaborating at the computer encouraged one another to invest more effort and tended to gain a higher level of mathematics achievement than their counterparts learning individually with CAI. These findings incorporate in previous studies showing the effects of cooperative learning on mathematics achievement and time-on-task (e.g., Mevarech, 1985; Stallings and Stipek, 1986). According to Salomon (1983), cognitive effects of media depends on a number of factors including the effort invested, depth of processing, and special aptitudes of individual learners. Future research may focus on these factors and relate them to learning mathematics cooperatively and individually in CAI settings.

REFERENCES

Jackson, A., Fletcher, B. C., and Messer, D. J. (1986). A survey of microcomputer use and provision in primary schools. Journal of Computer Assisted Learning, 2, 45-55.

Johnson, R., Johnson, D., and Stanne, M.B. (1985). Effects of cooperative, competitive, and individualized goal structures on computer-assisted instruction. Journal of Educational Psychology, 77, 668-677.

Kulik, J.A., Bangert, R.L., and Williams, G.W. (1983). Effects of computer-based teaching on secondary school students. Journal of Educational Psychology, 75, 19-26.

Mevarech, Z. R. (1985). The effects of cooperative mastery learning strategies on mathematics achievement. Journal of Educational Research, 78, 372-377.

Mevarech, Z. R. (1985). Computer-assisted different instructional methods: A factorial experiment within mathematics disadvantaged classrooms. Journal of Experimental Education, 54, 22-27.

Mevarech, Z. R. and Rich, Y. (1985). effects of computer assisted instruction on disadvantaged pupils' cognitive and affective outcomes. Journal of Educational Research, 79, 5-11.

Mevarech, Z. R. and Ben-Artzi, S. (1987). Effects of CAI with fixed and adaptive feedback on children's mathematics anxiety and achievement. Journal of Experimental Education, 56, 42-47.

Mevarech, Z. R., Stern, D., and Levita, I. (1987). To cooperate or not to cooperate in CAI: That is the question. Journal of Educational Research, 80, 164-168.

Osin, L. (1981). Computer-assisted instruction in arithmetic in Israeli disadvantaged schools. In R. Lewis, and D. Tagg (Eds.). Computers in education. Amsterdam. North-Holland Publishing Company. IFIP.

Salomon, G. (1983). The differential investment of mental effort in learning from different sources. Educational Psychologist, 18, 42-50.

Sharan, S. (1980). Cooperative learning in small group: Recent methods and effects on achievement, attitudes, and ethnic relations. Review of Educational Research, 5, 241-271.

Slavin, R. (1980). cooperative learning. Review of Educational Research, 50, 315-342.

Stallings, J.A. and Stipek, D. (1986). Research on early childhood and elementary school teaching programs. In M.C. Wittrock (Ed.) Handbook of research on teaching. NY: Macmillan Publishing Company.

Webb, N.M. (1982). Student interaction and learning in small groups. Review of Educational Research, 52, 421-445.

MATHEMATICAL PATTERN-FINDING IN ELLEMENTARY SCHOOL
-- FOCUS ON PUPILS' STRATEGIES AND DIFFICULTIES IN PROBLEM-SOLVING --

Nobuhiko NOHDA, University of Tsukuba, Japan

(Abstract)

Our study 'Problem-Solving' is the current focus on mathematics education in Japan. The study on analyzing pupils' strategies and difficulties in problem solving is considered indispensable to improve teaching in mathematics classroom activities. It seems that these strategies and difficulties are influenced greatly by some social and cultural factors, such as languages, symbols and daily life-habits etc.. This study is planned in order to make exactly the effects of social and cultural background on teacher and pupils who engage in problem solving by means of activities and communications, particularly in reference to share meaning and use of mathematical words and symbols involved in problem solving. We have to become more aware of the information processes which consist in the communications between the teacher's explanations and pupil's understandings about problem-solving.

Subjects of survey test in this study are selected at random one class of first, second and third graders in the elementary school and they are living in Tsukuba City near Tokyo. And then, we will take the second class for the problem-solving of the teaching experiment. The second-grade class (Male: 17, Female: 18, Total;35) we take here in this study, are composed of pupils of another elementary school which we have carried out the above survey test, but the school is the almost same conditions as the survey school in Tsukuba City.

1. Background Research

The process of problem-solving becomes evident when teaching is seen as a process of interaction between the teacher and learner-and among the learners-in which the teacher attempts to provide lerners with access to mathematical thinking in accordance with given problem. This teaching/learning process is (like all processes between learners) influenced by a number of social and developmental aspects and factors which can be included in problem-solving. The communication between teacher and learner is thus not only conditioned by formal decisions about goals, content and teaching methods, but it is also strongly dependent on even more informal aspects in early elementary school, such as the teacher's words and explanations to the problem-solving, and learner's motivations to solve the problem and to concern with it.

We will cite an example as the problem-solving activities between teacher and learners (Fig.3). A brief consideration of some of the roles of the teacher at different stages of the teaching/learning process illustrates this: instructor as to teach learner mathematical knowledges and skills (Top-Down); educator as to make them help problem-solving (Bottom-Up); and decision making as to judge teaching goes ahead or not, still there repeatly explaines more politely. The teacher's explication of such roles is intgrated with his specific actions, and serves to establish his/her background and context for the interactions between pupils' actual and inner activities in connection with any their subjective words.

If you grant this inherent subjectivity of concepts and, therefore, of meaning , you are immediately up against a serious problem. If the meanings of words are, indeed, our own subjective construction, how can possibly communicate? How could anyone be confident that the representations call up in the mind of the listener are at all like the representations the speaker had in mind when he or the uttered the paticular words? This question goes to the very heart of the problem of communications about problem-solving.

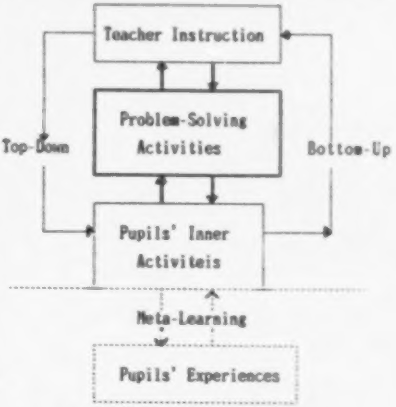


Fig.1 Problem-Solving Activities

Accordingly, the communication used 'problem-solving' as an organizing principle in Japanese mathematics learning calls for meta-learning under the teacher's support. This communication views mathematics classroom teaching as controlling the organisation and dynamics of the classroom for the purposes of sharing and developing mathematical thinking.

2. Mathematical Problem-Solving in Lower Elementary School

Our study 'Problem-Solving' is the current focus on mathematics education in our world. The study on analyzing pupils' strategies and difficulties in problem solving is considered indispensable to improve teaching in mathematics classroom activities. It seems that these strategies and difficulties are influenced greatly by some social and cultural factors, such as languages, symbols and daily life-habits etc.. This study is planned in order to make exactly the effects of social and cultural background on teacher and pupils who engage in problem solving by means of activities and communications, particularly in reference to share meaning and use of mathematical words and symbols involved in problem solving. We hope to become more aware of the information processes which consist in the communications between the teacher's explanations and pupil's understandings about problem-solving, i.e. pupil's hearing and writing down some key word the teacher says.

Here we use as non-routine problems: problem situations (Christiansen & Walter, 1986). We suppose the given problem by the teacher is caused for major difficulty: How to give a suitable problem to pupils ? In actual practice, every teacher will have to take his or her own classroom conditions into consideration. Thus, we will define the problem being used in this paper as follows: the problem includes both sides of mathematics and pupil/pupils, and then it is a non-routine problem which two fundamental factors must be contained in the problem in order to solve by themselves independentlly in mathematics classroom (Nohda, 1983, 1986).

Here we use the problem of pattern-finding. We shall focus on mathematical pattern-finding in problem solving. One of the dominant themes of cognitive research into problem-solving in recent years has been pattern-finding. However, much of this research has been in non-mathematical contexts (Lester, 1982). We will study pupils' acheivement on solving-problems from views of mathematics education. Thus,we will define the problem as follows: The problem occurs when pupils are confronted with a task which is usually given by the teacher and there is no prescribed way of solving the problem. It is generally not a problem that can be immediately solved by the pupils.

Pupils are able to solve the problem when they find a suitable 'pattern' in the problem. On the other hand, they have some feelings of difficulty in solving their problems when not being able to find a suitable 'pattern'. To study pupils'mathematical activities by means of the strategies and difficulties of problem-solving, is to make it clear how pupils find more suitable patterns of the problems under some interaction between the teacher and pupils, and between pupils, what strategies they find in their problem-solving, and in what parts they have difficulties in teaching and learning processes(Silver, 1979).

For the purpose of this study, first of all, we consider the mathematical activities through the following two cases. The one is the underlying pattern in the problem, that is, the nature of characterizing the problem itself. The other is the feature of strategies in pupils' problem-solving. The former means the structure of problem and the rule in it etc.. The latter is the mode of action applied in pupils' problem-solving. Therefore, in order that pupils might do better in their problem-solving, it is necessary that pupils share the understandings of problem through some activities of communications between teacher and pupils. For pupils who fail to understand the problem or feel difficult to solve it, the reason would be that there is no sharing the understanding or way of solving of the task through the interactions between tasks and pupils under the teacher's instruction.

To make clear the pupils' strategies and difficulties on problem-solving from the above viewpoints, we will present a more difficult problem than the problems found in the textbooks. Then, we will observe the mode of action on pupils' problem-solving and analyze the process of problem-solving which pupils take to solve the problem, and whether they arrive at the correct final answer or not. In studying pupils' pattern-finding behaviors, we may be able to see better how pupils are solving the problem and examine the steps by which they arrive at their understanding, planning, solving and checking by means of the interactions between teacher and pupils' communications(Polya, 1962). The interest here is to look at the internal thinking of pupils and to attempt to determine how their thinkings unfold by looking at their work on papers and to act and talk with the problem between the teacher and some pupils in the classroom by our observations.

3. Survey Test

Subjects in this study was selected at random one class of first, second and third graders in the elementary school and they were living in Tsukuba City near Tokyo. This test was carried out May 16, 1986 and that day was in a short time the biginning of new school year in Japan.

Survey procedures were that let the pupils read themselves the problem after the classroom teacher was handing the problem to pupils and then the problem out loud for all pupils to hear, and gave them 15 minites for solving the problem.

Survey Problem

Apple problem (See Figure 2)

(1) How many apples are there in this figure? (Count the number without skipping any and without counting any apple more than once.)

(2) Show different ways of counting the apples. How many different ways of counting can you think of?

(3) Of all your ways of counting, mark the one you think the best.

The feature of this problem's pattern is to take those as two pairs of apples forming with 5 × 5 row. That is, the pattern is 2 × 5× 5 here. Namely, the apples arranged with 5× 5 can be taken as those set as to pile up with shiftings lightly. Therefore, pupils found the same number and rule (pattern) in group of such formations as the case of Figure 3.

Figure 2 Apple Problem

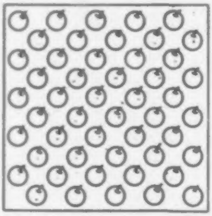


Figure 3 Problem Pattern

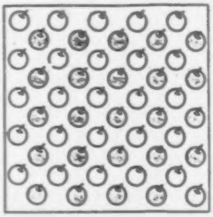


Table 1 shows only the result of the survey test item (3).

Table 1. Result of Survey Problem

A. Correct Counting	Grade First			Grade Second			Grade Third		
	Male	Female	Total	Male	Female	Total	Male	Female	Total
Numbers	17	17	34	15	23	38	17	17	34
Correct	9	2	11	8	9	17	13	13	26
No Responce	5	7	12	2	9	11	1	0	1

B. Ways of Counting

One By One	12(9)	10(2)	22(11)	5(4)	8(4)	13(8)	2(0)	7(7)	9(7)
Pairs			0	1(0)	2(0)	3(0)	2(2)	2(2)	4(4)
Group of Five			0	1(0)	0	2(0)	9(8)	4(2)	13(10)
Group of Ten			0	1(1)	2(1)	3(2)	3(3)	3(2)	6(5)
Aslant			0	4(2)	5(3)	9(5)	0	1(0)	1(0)
The Others			0	1(1)	1(1)	2(2)	0	0	0

Note: ( ) in parentheses in the Table 1 are those pupils of correct answers

The difficulty of this problem lies in that a first glance the pupils feel it rather difficult to count well because of seemingly complicated problem for the pupils of the lower elementary school. Especially, for first grade-pupils, it is difficult for them to count well after arranginig and regrouping in the same number of those or in the concept of pairs, although it is easy for pupils to count the number of up to fifty with numeral. They gain almost the same numbers as the correct answers, errors and no responses.

For second grade-pupils, it is easy for them to count apples being arranged and regrouped with the concept of pairs, group of fives and tens, and aslant. About half pupils gain the correct answers almost using one by one counting. They could not almost aquire the well-counting as group of fives or tens.

For third grade-pupils, pupils almost gain the correct answers and two thirds pupils are the well-counting after the arranging and regrouping with the concept of pairs, group of fives and tens.

Problem-solving we are concerned with here, is needed to share and develop of mathematical patterns as well-counting of the arranging and regrouping according to the concept of pairs, group of fives or tens on second grade-pupils. To study our problem-solving through the teaching experiment, we cannot take the first grade-pupils sharing and developing of mathematical pattern as the well-counting in this case, and need not teach the third grade-pupils the well-counting of apples. Thus, we will select the second grade-pupils for our study and to take the second grade class for the teaching experiment.

3.Problem-solving in the second-grade classroom

The second-grade class (Male; 17, Female; 18, Total;35) we took here in this study, were composed of pupils of another elementary school which we had carried out the above survey test, but the school was the almost same conditions as the survey school in Tsukuba City. This lesson was done June 6, 1986.

A classroom teacher started as follows: pupils were each given a picture of 'apples' which was a larger picture than usual one and put the same picture of 'apples' on the blackboard, and then, the teacher asked the pupils "How many apples are there in this figure?" and explained some notinons to them; "Counting the number under well-consideration without leaving some out or counting doble." After he explained to them the problem, he wrote the same informations about it on the blackboard as follows:

What way of counting and how many ways of counting do you think of?

Of them all, encircle your way of counting as you think good and suitable in this problem.

Pupils wrote their answers on the served sheets for about ten minutes after teacher explained the problem. While the teacher were observing and looking through pupils' activities of solving the problem in details, he advised first, some of pupil to take care of counting, and next, made them respectively to think out more ways of counting, and then he found out their different ways of the solutions as follows:

(1) Pupils almost were checking and counting apples with one by one vertically or holizontally, or with the filled numerals in the sketch of each apple. Some pupils mistook to count apples in their processes in this case.

(2) One fifth pupils were counting apples in pairs and some pupils who counted 2, 4, 6, 8, and so on continued to add apples till fifty, and the pupils almost gained the correct solution but a few pupils had the results in the impossibility of calculating 2 × 25 in this case.

(3) Four pupils who counted five apples together counted accurately and relatively quickly in this case.

(4) Nine pupils who counted ten apples together well-counted correctly and quickly in this case.

(5) A rather small number of the pupils used a symmetry of figure as the way of aslant counting. In this case, adding numbers aslant was the key.

Note: Almost all pupils made counting by more than about one method in this lesson.

4. Pupils' activities in the classroom

When pupils almost had finished to count and check the apples by themselves, first, the teacher asked them whether they counted the apple correctly. This was a beginning of a communication through the interactions between the teacher and the pupils for the sharing of the correct answer. This was an important point that the teacher judged his teaching on ahead or not. This decision making of the lesson was important rolles of the teacher. Then, after the teacher explained pupils to select to better one commensurate with their countings, he pointed out a representative pupil respectively of the five cases above mentined and let them explain of their ideas according case from (1) to (5) cases at the front place beside the teacher of the classroom.

In the case of (1), when a girl explained her idea, almost all pupils nodded to show that they agreed and understood one by one counting. There was an existance of the correct counting between the pupils in the classroom. Furthermore, the teacher advised a few pupils who could not count them correctly, made them to count again more careful. Thus, all pupils gained the corect answer and felt to satisfy with their needs to solve the problem. These processes of teaching and learning activities were the important communication for the aims of solving the problem in cooperation with the teacher and the pupils.

In the case of (2), when a boy explained his idea, pupils almost understood the count of apples in pairs. There was the existance of the sharing of the couning between the pupils. And then, the teacher advised the others pupils who could not add them correctly, to add again more careful. Thus, all pupils had the feeling of satisfications, too. These processes of activities were the meaningful commnication between the teacher and the pupils for the aims of mathematical solving the problem. Furthermore, Some pupils replied the case (2) when the teacher asked them "Which is better method of the counting apples between the case (1) and (2)?". This was more advanced negotiation because of his asking to make their counting with mathematical views.

In the case of (3), when the other girl explained her idea, many pupils easily understood her explanation and appreciated it. And in the case (4) pupils appreciated the good explain by an excellent boy. We were impressed what pupils had understood the mathematical patterns could be attained through the processes of their commnications. All pupils had appreciated the grouping of fives and tens of mathematical pattern of the problem, and the teacher did not need to expalain the best ways of counting more details. They thought out themselves the best ways of counting from their communication without the direction of teacher.

In the case of (5), when a fanny boy explained his idea, most pupils seemed to reject it. For most pupils felt it troublesome to count the numbers and to calculate the numbers in adding. This was another important aspect for their communication, because they could find easily the ways of counting as the case (3) and (4), and did not really conceive that 1 +3 +5 +7 +9 could be calculated as rather easily (1 +9) +(3 +7) +5.

5. Disscussion on mathematical problem-solving

Seen the strategies of solving here, first almost pupils take them one by one counting and next some pupils find the same number (pattern) in grouping of the length-width formation and a few pupils take the slanting formation. Many pupils find the same number and rule (pattern) in group of such formation. The difficult points here that in spite of the first instruction by the teacher, about half of the pupils counted twice and forgot some numbers to count. For the purpose of overcoming these difficulties, the mathematical ideas of grouping have developed by the human race for a long time ago.

We have to need the communication between the teacher and the pupils as follws: the teacher advises pupils who can not find the correct patterns, to find the features of problem and to count again apples using the ideas of grouping more careful. Thus, almost pupils understand the ways of counting from the cases (1) to (2), or from (2) to (3), or from (3) to (4) except case (5). Under the teacher's direction, they have the feeling of satisfications to learn new ideas and concepts in the mathematics lessons. A series of these communications open to the interactions betweem the teacher and the pupils for the main aims of solving the problem.

Furthermore, In the case of (3) and (4), for examples, we are impressed that pupils have the real appreciations of sharing of mathematical patterns by the processes of commnications between pupils by themselves. Because, all pupils have appreciated the grouping of fives or tens for counting the apples. They think out themselves the best ways of counting the apples with their communications without the directions of teacher. This is the most advanced communications, because the best counting of grouping which is developed mathematically by the human race, is found by their learnings.

6. References

(1) Christansen, B. & Walter, G. (1986), 'Task and Activity', Christansen, B et al (Eds.) "Perspective on Mathematics Education". D. Reidel Publishing Co.

(2) Von Glasersfeld, E. (1987), 'Lerning as a Cognitive Activety': Janvier C. (Ed.), "Problems of Representation in the Teachinf and Learning of Mathematics", Lawrence Erlbaum Associates.

THE CONSTRUCTION OF AN ALGEBRAIC CONCEPT THROUGH CONFLICT

Alwyn Olivier
University of Stellenbosch

This paper focuses on one aspect of pupils' interpretation of literal symbols in elementary algebra (generalized arithmetic), namely that different literal symbols necessarily represent different values. The underlying causes for the misconceptions are investigated. A teaching experiment inducing conflict and reflection to remediate the misconception is described and analyzed.

From a constructivist point of view, students' misconceptions are never arbitrary or altogether unreasonable. Misconceptions are seen as emerging from some interaction between experience and other existing concepts the student has (perhaps themselves misconceptions). Misconceptions are crucially important to teaching and learning for at least two reasons:

- misconceptions form part of the student's conceptual structure that will influence further learning, mostly in a negative way, because misconceptions generate mistakes.
- misconceptions are highly persistent and resistant to change through instruction. They are maintained by their ability to distort or reject incompatible information and by the support from other concepts in the student's conceptual structure.

In this paper we focus on the often-observed and well-documented misconception concerning the meaning of literal symbols in elementary algebra (generalized arithmetic), namely that different literal symbols necessarily represent different values (Küchemann, 1981; Booth, 1984a). A student's response of "never" to the following question usually demonstrates this misconception (Küchemann, 1981):

When is the following true – always, never or sometimes?
$L + M + N = L + P + N$

The pervasiveness of this misconception is illustrated by the following data for the question above: In the CSMS study (Küchemann, 1980) 56% and in our study involving more than 40 000 students (Olivier, 1988) 74% of 13 year olds answered "never". The resistance of this misconception to change is illustrated by the poor improvement in performance by average students in the SESM project, despite a well-designed teaching programme that was successful in ameliorating other algebraic misconceptions (Booth 1984a). Booth (1984b) attributes the persistence of this misconception to maturation-linked cognitive factors, i.e. that understanding depends on the attainment of a certain general developmental cognitive level. Küchemann (1981) links understanding of generalized number to Piaget's late-formal stage of development. However, the possibility remains that certain experiences (instructional interventions) may well address this misconception successfully, disproving the developmental hypothesis. For instance, Sutherland (1987), in studying students' understanding of variables in algebra in a Logo environment, concludes that "Whether or not pupils can make the links between variable in Logo and variable in algebra appears to depend more on the nature and extent of their Logo experience than on any other factor."(p.241) This promising approach nevertheless showed limited success in relation to the misconception under discussion.

SOME EXPLANATIONS

Searching for underlying causes of this misconception, we conducted interviews with ten students randomly chosen from the eighth grade population of semi-urban first-world secondary schools. Each interview was based on a subset of the following questions:

1. When is the following true – always, never or sometimes?

   $$L + M + N = L + P + N$$

2. If $a + b = 4$, what values of a and b will make the sentence true?
3. True or false: If $2a + 3b = 20$, then $a = 4$ and $b = 4$ is a solution of the equation.
4. Solve for x and y:

   $$\begin{aligned} x + y &= 6 \\ 2x + y &= 9 \end{aligned}$$

5. Construct an algebraic expression for the total points scored by a team in a rugby match if they scored only tries (counting 4 points each) and penalties (counting 3 points each). Use the expression to find the total points if a team scores

   (a) 5 tries and 2 penalties (b) 3 tries and 3 penalties

The following is a summary of the findings of the interviews and a situational analysis. All students interviewed demonstrated the misconception in questions 1 and 2.

For four students answering "never" in question 1, the literal symbols did not represent numbers, but names of objects like apples and bananas or abbreviations for names of objects (e.g. a stands for apple) or as an object in its own right (letters of the alphabet). These students are simply continuing their arithmetic framework of knowledge (Booth, 1984b), where literal symbols are often used as abbreviations for units (e.g. 4 m). Also, in introducing algebra, teachers often do not distinguish between symbols and their referents or use objects (apples and bananas) as referents instead of numbers to facilitate mechanical manipulation and inhibit conjoining (e.g. $a + b = ab$).

A further four students viewed the literal symbols in question 1 as representing unique, unknown values, from which it then follows that different symbols necessarily represent different values. This mind set may be established through the early emphasis on linear equations in the curriculum. It is also the outcome of experience. One student mentioned that he had "never, ever" seen different literal symbols stand for the same number (he was referring to substitution exercises of the type

"If $a = 2$ and $b = 3$ evaluate (1) ab (2) $a + 2b$ etc").

Despite their handling literal symbols as objects or unique unknowns in question 1, all ten students accepted more than one replacement of values in question 2, although no student admitted $a = 2$ and $b = 2$, even on being prompted on the possibility. They were all quite adamant about that. Two reasons were identified.

One reason is that pupils, despite working with numbers, do not seem to work with numbers in an abstract sense, but, to give meaning to the situation, introduce their own concrete referents for the literal symbols (e.g. "things" or apples and bananas) by reversing the modelling process. The following extract illustrates the point:

(Interviewer: I; Frieda: F explaining why $a \neq b$ in question 3)

F: I do not know in what circumstances the equation was asked. But if the *a* is the abbreviation for the apples and the *b* of the bananas, they must have different symbols.

I: So *a* and *b* are abbreviations for the apples and bananas.

F: No, I would rather say it's a symbol for the apples and bananas.

I: A symbol for the apples?

F: Yes, that you use to indicate what each number is. If you say *a* is equal to 4, then you know that if *a* is the symbol for apples, then you will immediately know that *a* stands for the apples and that 4 apples were bought.

Frieda's conceptualization should allow her to buy an equal number of apples and bananas, but her verbalizing "apples" instead of "number of apples" means that in the end the meaning degenerates to "apples" and "bananas", objects which should be different.

The second underlying mechanism for not allowing $a = b = 2$ as a solution to question 2, and in general not allowing different literal symbols to take equal values, stems from a combination of other valid knowledge and students' faulty logical inferences. Students are very much aware of the convention that the same literal symbol in the same expression must take the same value, e.g. in $x + 2x$. From this they infer that the converse, or even the inverse also holds:

- *Proposition*: the same letter stands for the same number.
- *Converse*: the same number stands for the same letter.
- *Inverse*: "Not the same letter" stands for "not the same number".

The following two extracts illustrate the converse reasoning to questions 1 and 2 respectively:

> "If they (M and P) were the same, you could just as well have used L + P instead of L + M. If P and M were the same number, then you cannot have P and M, because M and P represent different numbers, but if they (P and M) were the same number, it is the same letter that is used."
>
> "2 plus...no it cannot be...otherwise it would be $x + x = 4$. (silence). 2 plus 2 is 4, then you cannot have 2 plus 2, because it is the same numbers and it must be different numbers."

Another interesting phenomenon is that all students demonstrated the misconception in questions 1 and 2, while all students supplied correct responses to questions 4 and 5. Also of interest is that no student noticed any contradiction in their responses to the different contexts.

Lawler's (1981) theory of microworlds (cognitive structures) may offer an explanation. Students are operating in different distinct and separate microworlds when solving the two classes of problems. Lawler views the microworlds as actively competing with each other, working in parallel to solve a problem. Which microworld provides an answer to a problem depends on how the problem is posed and the particular knowledge the different microworlds embody. The competition of microworlds usually leads to the dominance of one and the suppression of others. To Lawler, resolving the misconception requires the cooperation, interaction and integration of microworlds whereby confusion between related competing microworlds is suppressed by a new control structure.

Davis (1984) also suggests that separate, conflicting "frames" may be created. A frame acquired early and developed well may prove to be extremely persistent, so much so that it may sometimes continue to be retrieved inappropriately long after one has become fully cognizant of the conditions under which it is or is not used. Put differently: a new appropriate frame may be available, but the old frame continues to exist. The source of such misconceptions lies in retrieving the wrong frame and not recognizing the retrieval error. As for remediating the misconception, Davis advocates making sure that pupils are aware of both frames, and of the likelihood of incorrect choice.

From our analysis of the data it is clear that most pupils possess two apparently separate schemas for literal symbols. One is the letter-as-object schema, which stresses the difference of different letters and which is appropriately used to make routine manipulation of symbols automatic (Skemp, 1971). The other is the letter-as-generalized-number schema, which should include the possibility that different literal symbols can take the same value. The essence of the observed misconception lies in the fact that the letter-as-object schema is inappropriately invoked in cases were it does not apply. As such the letter-as-object schema has become an obstacle to further learning, inhibiting the letter-as-generalized-number schema.

A TEACHING EXPERIMENT

For the purposes of a teaching experiment the format of the interviews was changed, by confronting students with the contradictions in their responses (questions 1 and 2 versus questions 4 and 5), in an effort to induce cognitive conflict and to help students to reflect on their own concepts and mental processes. The objectives of the further investigation were:

- to determine the strength and stability of students' beliefs concerning the misconception, and
- to determine the success of cognitive conflict as a teaching strategy to remediate the observed misconception.

An additional 30 students were interviewed. Of these, 22 demonstrated the misconception in questions 1 and 2, with correct responses to questions 4 and/or 5, before being confronted with the anomaly in their responses. After these confrontations the students were evenly split between

- persistence in the misconception
- total confusion
- successful remediation.

In the first category of students, the belief in the misconception was so strong that, on being confronted with the discrepancy in their responses, they chose to alter their initial correct responses by also excluding equal values in questions 3, 4 and 5, or reconciled the discrepancy by inventing all kinds of conditions for equal values in questions 3, 4 and 5, in preference to modifying the misconception and allowing equal values in questions 1 and 2. For example, Jacques, on comparing his response to question 4 ($x = y = 3$) with his response in question 2 (where he insisted $a \neq b$):

> "You can say that $x = y$, because you *proved* that $x = y$. See, you have proved that $x = y$. But here (question 2) nothing is proved yet, so you cannot say that $a = b$."

Similarly, students defended equal values in question 3 and 5(b) "because they say so", but excluded $a = b$ in question 2.

The second group of pupils typically obtained equal values in questions 3, 4 and 5. Then, when their attention was drawn to the fact that they would not allow equal values in questions 1 and 2, they altered their responses to questions 3, 4 and 5, only to be convinced again that equal values were common sense in 5(b), yet they would not accept equal values in 1 and 2. At that stage they were totally bewildered and confused.

Consider Thys as an example. After successfully completing 5(b), his expression was $4a + 3b$, he was asked why he did not allow equal values in question 2.

T: Oh no! Yes...it cannot be the same! (referring to 5(b)). It cannot be the same...I thought...I'm afraid I, I...(silence).

I: What was *a*?

T: *a*...*a* was 3.

I: What was *b*?

T: Also 3.

I: Can they be?

T: No!

I: But we just did a problem were they can!

T: Yes, but, but then that should have been an *a* (points at *b*)

I: What does *b* stand for?

T: The penalties.

I: And *a* stands for the ...

T: It's the tries.

I: Can we score three tries?

T: Yes.

I: Then *a* is 3?

T: Yes.

I: Can we score three penalties?

T: Yes.

I: Then *b* is 3?

T: Er... No, it must be an *a*.

I: Can the expression that we must write for the team's total, can it be $4a + 3a$?

T: Yes.

I: What does the *a* stand for?

T: It is the tries.

I: And what about the penalties?

T: It must also be *a*... oh no! (silence). I'm afraid I must now be totally confused ... (silence) ... No, I don't know.

The third group of students successfully altered their misconception responses for questions 1 and 2. They were all able to re-interpret the letters in these questions as letters with added semantic meaning (Rosnick, 1982), i.e. letters that mean more than a number – they mean a number of things. Carl, for example, after completing 5(b) and being confronted with the discrepant meanings: "Oh, so *a* can be the number of tries and *b* can be the number of penalties and a team could have scored 4 times" (question 2).

DISCUSSION

In summary, one-third of the students interviewed did not experience conflict between their discrepant meanings of literal symbols. Another one-third of the students experienced the conflict quite emotionally, and although they were confused and unable to resolve the conflict, it is possible that they may do so with more experience and/or reflection. Although the other one-third of the students were successful in the interview situation, it is of course not claimed that the changed perspective was permanent. It was not possible to follow up any of the cases.

The relative ease with which the successful group seemed to correct their misconception would suggest that the cognitive structures necessary for such assimilation were already available to the students. It is suggested that the main factors distinguishing successful students are the *absence* of the converse-flaw and *avoidance* of the letter-as-object trap. The teaching experiment did not address the converse-flaw. It is suggested that situations involving semantically laden letters have a constructive role to play in resolving the misconception, because they render equal values for different letters intelligible and help to form a bridge between the meaning of letters in language and its meaning in mathematics. The complexity of the pitfalls in language when viewing letters as objects is depicted in Fig. 1 (compare the extracts for Thys and Frieda). Fig. 2 shows the simplicity of a correct interpretation of semantically laden letters. Students who view letters as objects must negotiate more transformations and make more errors. It was observed that unsuccessful students introduced objects even in abstract numerical problems.

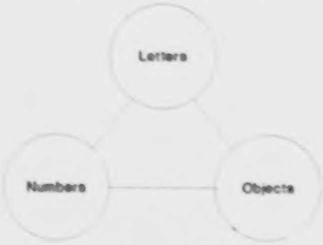


Figure 1

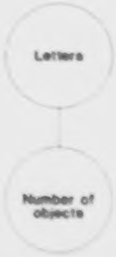


Figure 2

REFERENCES

Booth, L.R. (1984a). **Algebra: Children's Strategies and Errors.** Windsor, England: NFEF-Nelson.

Booth, L.R. (1984b). Misconceptions Leading to Error in Elementary Algebra. **Journal of Structural Learning, 8**, 125-138.

Davis, R.B. (1984). **Learning Mathematics. The Cognitive Science Approach to Mathematics Education.** London: Croom Helm.

Küchemann, D.E. (1980). **The Understanding of generalized Arithmetic by Secondary School Children**. Unpublished doctoral dissertation, Chelsea College, University of London, London.

Küchemann, D.E. (1981). Algebra. In K. Hart (Ed.), **Children's Understanding of Mathematics: 11-16.** London: Murray.

Küchemann, D.E. (1982). Object Lessons in Algebra? **Mathematics Teaching**, March 1982, 47-51.

Lawler, R.W. (1982). The Progressive Construction of Mind. **Cognitive Science, 5**, 1-30.

Olivier, A.I. (1988). **Leerlinge se begryping van lettersimbole in elementêre algebra** [Students' Understanding of Literal Symbols in Elementary Algebra]. Unpublished doctoral dissertation, University of Stellenbosch, Stellenbosch.

Rosnick, P.C. (1982). **The Use of Letters in Precalculus Algebra.** Unpublished doctoral dissertation, University of Massachusetts, Massachusetts.

Skemp, R.R. (1971). **The Psychology of Learning Mathematics.** Middlesex, England: Penguin Books.

Sutherland, R. (1987). A study of the Use and Understanding of Algebra Related Concepts within a Logo Environment. In J.C. Bergeron, N. Herscovics & C. Kieran (Eds.), **Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education.** Montreal.

GENDER AND MATHEMATICS: THE PREDICTION OF CHOICE AND ACHIEVEMENT

WILMA OTTEN & HANS KUYPER

RION: INSTITUTE FOR EDUCATIONAL RESEARCH
UNIVERSITY OF GRONINGEN

ABSTRACT

This paper deals with gender differences in the prediction of 1) the choice of math as an examination subject, and 2) the achievement in math. Predictors were gender, attitude towards math, whether favored vocational training requires math and optionally achievement and choice. The attitude was assessed by two approaches: scale-construction and the Fishbein model. Multiple regression analysis showed that more than 70% of the variance in math choice could be predicted against 50% in achievement. Gender differences were profound in the prediction of math choice. These differences could be attributed to gender differences in favored vocational trainings.

INTRODUCTION

One examination subject has the special attention of the Dutch government, namely: mathematics. Mathematics is considered to be important. It is required for most vocational trainings and the consequential professions generally are less struck by unemployment than those which don't require math. Based upon these facts the government has started a national propaganda campaign "Choose exact sciences". Another reason was given by the fact that generally more boys than girls choose mathematics. So girls are likely to decrease their chance of finding jobs due to their choice of examination subjects.

In this paper the choice of mathematics as an examination subject is the main topic. Which are the main predictors of the decision to choose mathematics as an examination subject and do boys differ from girls in this respect? This question raises five variables of interest: the sex of the pupil (SEX), the choice of mathematics as an examination subject (CHOICE), whether mathematics is required for the favored vocational training (REQUIRE), the achievement in mathematics (ACHIEV) and the attitude towards mathematics (ATTITUDE). This attitude consists of several sets of items, which will be discussed in the method section. We assume that SEX, REQUIRE, ACHIEV and ATTITUDE influence the math choice, and are therefore predictors of CHOICE. The next question is whether the relation between REQUIRE, ACHIEV, ATTITUDE and CHOICE differs between the sexes. We acknowledge the existence of interrelations between the variables, but they are not our main interest. The second topic of this paper concentrates upon the prediction of achievement in mathematics. Specifically, which are the main predictors of the achievement in mathematics and do boys differ from girls in this respect?

METHOD

In May and June of 1986 the research was undertaken in general formative secondary schools of all three levels of difficulty. In this paper we concentrate on the results of the intermediate difficulty level. This kind of secondary school takes five years. In the third year the examination subjects are chosen. Therefore, in this paper we report mainly the results concerning pupils in the third year (age: 14, 15 years). The total number of pupils was 354; 210 girls and 144 boys. The pupils filled out questionnaires during a subject hour at their schools. The questionnaire contained a large number of variables, including the variables of our interest:

CHOICE: "Are you going to choose mathematics?" Answer possibilities: certainly not (1), probably not (2), do not know yet (3), probably (4), certainly (5);

REQUIRE: The pupils were asked to state their favored vocational training. If any, they indicated whether mathematics is a requirement for entering it;

ACHIEV: The pupils were asked to give the mathematics marks on their last two school-reports. The mean of both marks was used as achievement-index. The Dutch rating-system ranges from 1 (very poor) to 10 (excellent);

ATTITUDE: We adopted two different approaches to assess the attitude towards mathematics;

ATT I : Item analysis and scale construction.

In this approach three main attitude domains were distinguished:

a. Pupils' personal attitudes (23 items; 4 subscales)
b. Pupils'perception of math teacher's behavior (16 items; 3 subscales).
c. Perceived sex-role ideas of the math teacher (10 items);

ATT II: The Fishbein model (Ajzen & Fishbein, 1980).

The model distinguishes two components that influence the intention to perform (a) behavior: the 'attitude' towards the behavior and the 'subjective norm' about the behavior. The attitude-component consists of behavioral beliefs, i.e. expected consequences of the behavior, and evaluations of these beliefs. The subjective norm-component consists of normative beliefs, i.e. perception of the degree to which important others favor the behavior, and motivations to comply, i.e. the degree to which these perceptions are complied to.

After multiplication of the probability ratings (-3=certainly not, 3=certainly) by the importance ratings (1=very unimportant, 5=very important) of the behavioral beliefs twelve 'attitude'-components resulted:

qualifying for an education which requires math, qualifying for a profession which requires math, not being able to choose another examination subject, increase of professional possibilities, the time spent on math home-work, passing the examination at first try, need of additional lessons, increase of grade point average, kind of teacher ((un)friendly), kind of classmates ((un)friendly), (foster)parents' satisfaction.

Eight subjective norm-components resulted after multiplication of the probability ratings (that other person favored the math choice; -3=certainly not, 3=certainly) by the compliance ratings (1=no compliance, 6=much complicance):

(foster)father, (foster)mother, elder brother, elder sister, friend, math teacher, class mentor, schoolcounsellor.

For more details on this method, see Kuyper & Otten, 1988.

RESULTS

The variables of interest showed the following results. Of the boys, 81% intended to choose math (category 4+5 vs 1+2) versus 43% of the girls; 63% of the boys favored a vocational training which requires math, versus 21% of the girls. The mean math mark for the boys is 6.1 (sd=1.2) versus 5.9 for the girls' (sd=1.2).

The item analysis and scale construction approach consisted of principal component analysis followed by varimax rotation, and assessing Cronbach's alpha for items belonging to one factor (absolute loading $\geq$.50). Finally a scale-value resulted by calculating the mean of the scale items. Analyses of the items within each attitude domain resulted in the following eight scales:

'pleasure in math' (6 items, a=0.90); 'difficulty of math' (12 items, a=0.86); 'sex-specifity of math' (3 items, a=0.65; e.g. "girls don't need math"); 'usefulness of math' (2 items, a=0.71; e.g. "math is useful for society"); 'perceived knowledge transfer by teacher' (8 items, a=0.86; e.g. "encourages asking questions"); 'perceived relevance transfer by teacher (4 items, a=0.74; e.g. "tries to convince the pupils of the relevance of math for later life"); 'perceived sex-specific behavior of teacher' (4 items, a=0.66; e.g. "asks girls easier questions than boys"); 'perceived sex-role ideas of teacher' (10 items, a=0.87; e.g. "math is a subject for males").

To answer the two main questions of this paper we used the technique of multiple regression analysis. The inclusion of predictors in the equations was determined by stepwise selection (forward and backward elimination). Missing data were handled by the SPSS-X option pairwise deletion. In addition to an 'overall' analysis, seperate analyses for boys and girls were performed.

I Prediction of the math choice.

The predictors are SEX, REQUIRE, ACHIEV and ATT I. Of course SEX is excluded from the boys' and girls' analysis. Another regression analysis was performed analogous to the former except for replacing ATT I by ATT II.

The overall analysis, including ATT I, yields an R of .84 (70% of the variance in CHOICE accounted for). In the girls' analysis 67% of the variance is accounted for, in the boys' analysis 61%. The β-weights are an indication of the relative importance of the predictors. Table 1 displays the ß-weights of the predictors included in the equations.

Table 1: Multiple regression models for the prediction of CHOICE using as SEX, REQUIRE, ACHIEV and (ATT I) as predictors: R^2 and ß-weights.

	overall	girls	boys
R^2	.70	.67	.61
SEX	.08		
REQUIRE R	.40	.30	.62
ACHIEV	.15	.18	
ATT I:			
pleasure	.17	.22	
usefulness	.12	.15	
difficulty	-.22	-.23	-.31
knowledge transfer	-.08		

The main predictors are REQUIRE and difficulty. The negative ß-weight of difficulty indicates that the more difficult math is, the less math is chosen. The negative ß of knowledge transfer seems surprising, because it indicates the more knowledge transfer, the less math is chosen. However, this effect is due to the removal of the covariance between CHOICE and it's former predictors from the initial correlation between CHOICE and knowledge transfer ($r=.13$), resulting in a negative partial correlation coefficient ($r=-.07$). SEX is included in the overall equation indicating that, despite the contribution of the other predictors, SEX contributes to the prediction of CHOICE in such a way that more boys choose math. The differences between the girls' and boys' solution are the following. First, the boys' equation accounts for less variance in CHOICE than the girls' equation. Second, the boys' equation is more 'economic': only two predictors versus five girls' predictors. Third, the large influence of REQUIRE on the boys' choice is striking. Lastly, the girls also include the predictors pleasure and usefulness.

The overall equation, including ATT II, accounts for 76% of the variance in CHOICE. The girls' equation accounts for 72% and the boys' accounts for 64%. Table 2 shows the ß-weights of the predictors included in the resulting models.

Tabel 2: Multiple regression models for the prediction of CHOICE using SEX, REQUIRE, ACHIEV and (ATT II) as predictors: R^2 and ß-weights.

	overall	girls	boys
R^2	.76	.72	.64
REQUIRE	.29	.18	.46
ACHIEV	.09	.14	.21
ATT II:			
education	.09	.12	
no other subject	.09	.08	
future possibilities	.09	.09	.14
pass at first try	.15	.20	.28
extra lessons	.11		
kind of teacher	-.07	-.10	
parents' satisfaction		.10	
mother	.14		
friend	.10	.16	
math teacher	.15	.26	

The main predictors are REQUIRE, pass at first try and ACHIEV. In this analysis SEX is not included in the equation. The differences between the boys' and girls' equations are similar to the differences noted above. First, the boys' equation accounts for less variance in CHOICE and is more 'economic': four predictors versus ten girls' predictors. Second, the large influence of REQUIRE on the boys' choice is striking again. Third, the girls' model shows the inclusion of 'other person' predictors: friend, math teacher, parents' satisfaction and kind of teacher. The negative ß-weight of the last predictor originates from a negative initial- and partial correlation coefficient (initial r=-.03; partial r=-.10), indicating the negative influence of the attitude-component kind of teacher on the math choice.

II Prediction of the math achievement.

The criterion is ACHIEV and the predictors are SEX, CHOICE, REQUIRE and ATT I. SEX is excluded from the boys' and girls' solution. The results are shown in Table 3.

Table 3: Multiple regression models for the prediction of ACHIEV using SEX, REQUIRE, CHOICE and (ATT I) as predictors: R^2 and β-weights.

	overall	girls	boys
R^2	.52	.47	.58
SEX	-.18		
CHOICE	.26	.27	.19
ATT I:			
difficulty	-.56	-.47	-.64
sex-role idea teacher	.09		

The overall analysis yields an R of .72 (52% of the variance in ACHIEV accounted for). In the girls' analysis 47% of the variance is accounted for, in the boys' analysis 58% is accounted for.
The main predictors are difficulty and CHOICE. The negative β of difficulty indicates the more difficult math, the lower achievement. The initial correlation between ACHIEV and SEX is .05, which could be expected considering the mean math marks of the sexes (boys: m=6.1, girls: m=5.9). However, partialing out the covariance between ACHIEV and it's former predictors results in a partial correlation coefficient between SEX and ACHIEV of -.17, which explains the negative β-weight of SEX. Surprising is the inclusion of perceived sex-role idea of math teacher, indicating that the more sex-stereotyped opinions are attributed to the teacher, the lower the achievements are. There are no striking differences between the girls' and boys' models.

DISCUSSION

Returning to the first topic, prediction of the math choice and sex differences in this respect, we may conclude the following. First, it appeared that the choice of math could be predicted to a large extent. In both analyses (ATT I/II) more than 70% of the variance in math choice could be predicted. Second, the achievement in math, the attitude towards math and whether the favored vocational training requires math are significant predictors of the math choice. Third, sex proved to be a significant predictor for choice of math using the scales as predictor set. However, sex was not included in the regression equation when using the Fishbein predictor set. When the effects of these predictors were partialled out, sex was not significantly related with choice of math. Therefore it seems plausible that sex-effects can be explained by means of these predictors. The results of analyses, separately carried out for boys and girls, supports the above conclusion even more.

First, it seems that boys' choices are predominantly influenced by pragmatic factors such as difficulty of math, achievement in math, passing the examination at first try and especially whether the favored vocational training requires math, whereas girls' choices are also influenced by other persons and pleasure in math. It seems that whether or not the favored vocational training requires math determines the boys' choice of math above all. Girls' choice behavior is less prescribed by the conditions of the favored profession. Second, girls favor less vocational trainings requiring math. Third, no significant differencesin math achievement between the sexes was observed. Therefore we may conclude that the boys' preference for vocational trainings requiring math regulates their choice behavior in achieving this goal, whereas girls' preference for vocational trainings not requiring math allows their math choice to be influenced by other factors such as pleasure in math. This implies a more central role of the favored vocational trainings in further research after gender effects on math choice.

The second topic, prediction of math achievement and sex differences in this respect, leads to the following conclusions. Math achievement could not be predicted as well as math choice (about 50% of the variance accounted for). The choice of math and the difficulty of math appeared to be the significant predictors. Interesting is the absence of whether the favored vocational training requires math as predictor. Apparently this factor doesn't influence the math achievement, whereas it influences the math choice for a great deal. The low predictability of math achievement might be due to the absence of predictors like intelligence, mathematical ability, motivation and invested effort.

These predictors might also explain the inclusion of sex as a predictor, despite the absence of sex differences in achievement.

BIBLIOGRAPHY

Ajzen, I. & Fishbein, M. (1980). *Understanding attitudes and predicting social behavior*. Englewood Cliffs: Prentice Hall.

Kuyper, H. & Otten, W. (1988). *Gender differences in the choice of mathematics: a decisional approach*. Paper offered to The Twelth Annual Conference Of The International Group For The Psychology Of Mathematics Education (PME XII).

TEACHING AND LEARNING METHODS FOR PROBLEM-SOLVING : SOME THEORETICAL ISSUES AND PSYCHOLOGICAL HYPOTHESES.

Janine Rogalski, CNRS.
Aline Robert, Université Paris VI.

Abstract

Many researches have recently emphasized the role of metacognition in problem-solving. This paper focuses on methods as part of this field. Does it exist methods in problem-solving in a given mathematical field (geometry for instance)? What are the relationships between methods and classes of problems? Is it possible to teach methods? Can such a training be efficient for managing and/or acquiring conceptual knowledge? What problems and what didactical environments are good "candidates" for such a training? Does it exist an optimal moment in the process of knowledge acquisition for teaching methods? We insert these questions in a constructivist view of knowledge acquisition, and propose in this framework some psychological and didactical hypotheses based upon empirical studies.

Introduction

Metacognition has been studied from several points of view. Research on metacognitive development became an important element in cognitive psychology: how does "knowledge about knowledge" arise in child development, what role does it play in operational knowledge? This concerns a variety of cognitive activities, only a part of them being linked to problem-solving (Flavell, 1977). We want to underline the attempt presented by Pinard to develop a post-piagetian analysis of the origine of metacognitive knowledge and self-regulatory processes (Pinard, 1986). His study extends the question to a life-span perspective: it allows to take into account the problem of complex acquisitions such as mathematics, and scientific or professional knowledge. More specific research was engaged in psychology of mathematical education, concerning problem-solving. Most of them emphasized the positive role of metacognition in mathematical performance, through a theoretical analysis (Garofalo and Lester, 1985) and/or by analysing students strategies in problem-solving (Galbraith, 1985; Garofalo, Kroll and Lester, 1987). Some of these studies are directly concerned with the question of teaching competences for problem-solving (Schoenfeld, 1985; Garofalo and al., 1987). Otherwise, near questions arised in the field of artificial intelligence and education: how to integrate "reasoned explanations" and "reasoning on reasoning" is a crucial point in designing intelligent tutoring systems (Vivet, 1987). (1).

Our own present purpose is to specify some questions dealing with a specific area of metacognition : methods in mathematical problem-solving. First, we will precise what we mean by "methods" (by respect to students' heuristics or strategies and by respect to mathematical algorithms). We express some central theoretical issues concerning the status of methods in a given conceptual field: relationships between "local" acquisitions (knowing and knowing-how) and "global" organisation of problem-solving; relationships between a method and the "activation" of soon acquired knowledge. Secondly we present cognitive and didactical hypotheses: how and when to teach methods; what are the expected effects on knowledge acquisition and on knowledge "management". These hypotheses are based upon a theoretical analysis and upon empirical results in cognitive studies on decision-making and planning, on studies about teaching methods in other scientific or professional domains, and on detailed analysis of the role of teaching methods in geometry problem-solving with advanced level students.

Methods in problem-solving

A method is related to a class of problems. It expresses the commun points in efficient problem-solving in a field. It does not describe students' behavior. Roughly speaking, a method describes or even prescribes efficient ways in solving a given class of problems. It can be defined in terms of the functions fulfilled by respect to task requirements. A method can be considered as an invariant in problem-solving procedures linked to an invariant related to the class of problems (Robert, Rogalski, Samurçay, 1987; Rogalski, 1987).

Consequently, the specification of a method is related to the extend of the considered class. A method defining an organisation of phases in problem-solving as "problem understanding", "orientation", "organization", "execution", "control" is valid for whatever type of problems. It may precise a lot of "what" to do, but a few on "how" to perform it: how to analyse the problem, how to define the involved knowledge field, how to identify the possible strategies.. At the opposite the algorithm defining how to process with binomial equations has a very limited validity field. Henceforward, when speaking of methods, we will exclude pure algorithms and consider methods as presenting two main purposes: helping an user in the approach of a problem and in the organisation of the process leading to a solution (including control of solution validity and/or optimization) (2). As a class of problems may be embedded in a broader class, there exist embedded levels for methods, increasing the field of validity, and decreasing specifications about how to apply the method. On the other side, a given method may define sub-problems, for which it can precise methods of "lower" level (Rogalski M., 1987).

Methods for problem-solving in teaching and learning processes

One can constrast methods in problem-solving according to the following poles: methods which are strongly linked to conceptual invariants, and methods which are mainly devoted to organize, manage and control the use of soon acquired knowledge. An example of the first pole is given by programming methods for the construction of loops invariants in writing iterative programs. At the second pole one can find the methods implicit in heuristics management in expert systems. An example will be detailed below, which can be seen as an elicitation of expert's knowledge in the study of numerical sequences.

It appears a plausible hypothesis that methods play different roles in the teaching and learning process depending on their position by respect to these poles. As an example, we will now present "a priori analysis" of two methods, designed for scientific advanced level students. The first one deals with geometrical problem-solving; the second one with convergence of numerical real sequences.

The purpose in elaborating a method for complex geometrical problem solving was to teach them to students, so that they became able to conceive solutions to problems of the relatively large field covered by the curriculum in the scientific classes at the end of the secondary school (17-18 years old students). Requirements in writing proofs were out of this actual aim (Robert and al.). Schematically the method is organized in three parts: 1) a rough classification of types of geometrical problems (6 or 7 types), 2) a list of tools (such as: cartesian coordinates, transformations as symmetries, translations, rotations..., use of scalar product, barycenters..) with a specification of the setting in which they can be used (affine, vector or euclidian space, numerical setting..) and 3) a list of basic configurations (they are relatively simple configurations which appear very frequently in more complex figures, and whose properties are well known). This method was taught to students from the very beginning of the curse, according to the following scenario. Before any problem-solving situation, the teacher presented some of the above elements of the method. A completion of the initial state of the classification and the lists of tools and configurations was engaged by the students, depending of their activities in geometrical problem-solving. A great part of these activities was devoted to research in small groups (3 or 4 students) on problems requiring the use of the method: problems were given without any indications, several ways were possible to find a solution. The teacher intervened both on geometrical content and on methods. What was expected from the students was the following: asking questions about the type of problems, making suggestions about possible adapted tools, trying strategies and changing points of view, frameworks or strategies if unsuccessful. At the end of the work, the teacher presented a point on the various specific strategies uses in the different groups, and the geometrical concepts underlying the solution to the problem.

Writing a method for the study of numerical real sequences was done in a quite different perspective (Rogalski M., 1987). The method was not directly taught to students (in the teaching process) but was proposed to them after the curse (Students are scientific students, in the first year of the university). The purpose was to express a general, complete method for studying convergence for sequences frequently encountered in mathematics at this universitary level. The method was organized with strategies (more local) methods for reaching sub-goals): 1) classification strategy, 2) strategy for research of hypotheses (as: existence, possible value of the limit..), 3) proof strategy. Some of these strategies involve tactics (classify the problem, define priorities, simplify, modify for simplication, classify the sequence..); tactics themselves use technics (graphical representation, numerical tests...). Moreover a process was expressed for control, correction, "recovery" for dealing with unsuccessful strategies. At last, three types of required knowledge were presented, which have to be always available by students (consisting of main theorems, classical results and "standard" numerical functions). Two technics are joined as general useful tools: "using inequalities" and "reasoning by induction". This presentation was based upon students' previous knowledge and centered on the organization of the process of problem-solving. It clearly examplifies a method as a tool in managing soon acquired "local" knowledge (about specific sequences, typical problems such as convergence of sequences defined by induction..).

Some hypotheses and results about cognitive acquisition and didactical processes

Our hypotheses about the productive role of learning methods are based upon three types of considerations. First a constructivist conception of knowledge acquisition leads to the fundamental assumption that "problem solving is source and criterion of knowledge" (Vergnaud, 1982). Then, learning methods for problem solving should be strongly linked to knowledge development. Secondly, epistemological as empirical studies show that metacognition is an intrinsic part in the whole process of knowledge acquisition (Schoenfeld, 1985, 1987). Thirdly, studies in work psychology have shown strong evidence that goal setting (that is specifying goals to be reached in performing a task) has positive effect on the performance. (Locke, Shaw, Saari, Lathan, 1981). Now, methods organize research activity in problem-solving both by setting specific goals and relating sub-goals and tools, therefore they must lead to besser performance.

We can specify briefly two hypotheses about the process by which learning and using methods may improve knowledge acquisition. These hypotheses are based upon two theoretical concepts. First we defined the notion of "precursor": precursors for a new conceptual field are notions, operations and/or representations in a near field that can make new problems and notions meaningful. Second we defined twp states for student's knowledge: available and liberable. An avalaible knowledge can be used without any explicit cue in the problem, and without reference to this knowledge; at the opposite, a liberable knowledge requires an explicit call to this knowledge: specific goal directing attention to it, or specific cue in the text of the problem (such as are indications on the way by which solving a problem). Our hypothesis is that for most of the students existence of precursors is a strong requirement in acquiring new knowledge, and that knowledge has to become available in order to be really efficient in problem-solving (3).

Two hypotheses about teaching and learning methods are related to these concepts of precursors and states of knowledge: a) learning a method in a given conceptual field is more efficient (or even perhaps only possible) if there exist precursors for the involved conceptual notions and if some knowledge is present in "liberable" state; b) learning and using a method in problem-solving is a mean for a change in knowledge state, from "liberable" to "available", because of two facts calling out knowledge: elicitation of goals and explicit proposals of tools. Working in small groups may stress this productive role played by the elicitation processes.

A twofolds question arises at this point: what are the conditions for students'acquisition of methods? what are the conditions for teaching methods? The acquisition may follow an explicit presentation by the teacher (as in our first preceeding example on geometry) or it may proceed from elicitating students'strategies in problem-solving; the teacher expressing the invariants defining the method. The empirical results in professionnel acitivities as in teaching lead not to retain the hypothesis that students can coantruct themselves the invariants in eficient problem-solving: it concerns probably a small part of students, and it semms to us necessary to research pedagogical strategies for the others.

Depending on our preceeding psychological analysis we assume that 1) the possibility for students for acquiring a taught method depends on the content and on the actual state of knowledge by students; 2) didactical intervention is more efficient if methods are presented during problem-solving sessions, where students work in small groups and when the problems are "open" (no intermediate indications, and several ways for solution). The analysis of students working in small groups confirms the place devoted to elicitation, and its evolution along the successive sessions (Marilier, Robert, Tenaud, 1987).

Conclusion

Theoretical analysis, results in the field of cognitive psychology and data observed in didactical experiments converge to the conclusion 1)that one can design methods related to a specific conceptual field; 2)that such methods can be taught to students, as soon as they have some available knowledge and the ability to explicit metacognitive activities in a precise way, and to take them as objet for thought, and 3)that students benefit from such a teaching. Didactical situations which appear as good "candidates" for supporting such a methodological teaching involve: work in small groups, open and sufficiently complex problems and a didactical environment giving a large place to students' metacognitive activities such as discussion about knowledge and heuristics, and elicitation of metacogntive representations on mathematics, problem-solving, on learning and teaching maths (4). Two open questions concern to what extend such conclusions may be valid for teaching younger and less advanced level students, and what are the good ways for evaluating such a teaching and learning process?

Notes

(1) We don't try to be exhaustive, but to give some representative examples of different types of research in the field of metacognition in problem-solving (The first one devoted to mathematical problem-solving being Polya (Polya, 1962-64).

(2) The field of programming presents quite a wide range of "programming methods" one can analyse as methods for problem-solving (Rogalski, Samurçay, Hoc, 1987).

(3) From our theoretical point of view, these notions of precursors and liberable knowledge are related to Vygotski's concept of "proximal zone of development".

(4) The "beliefs" in Schoenfeld's classification (Schoenfeld, 1987).

Bibliography

Flavell J.H. (1977) Metacognitive development. in Structural/Process models of complex human behavior. J.M.Scandura and J.C. Brainerd (Eds). Sijthos and Noordhoss, Netherlands, 213-242.

Galbraith P.L. (1985) The use of mathematical strategies : factors and features affecting performance. Educational Studies in Mathematics, 413-441.

Garofalo J., Lester F.K. (1985) Metacognition, cognitive monitoring and behavioral performance. Journal for Research in Mathematics Education, 16,3,163-176.

Garofalo J., Kroll D.L., Lester F.K. (1987) Metacognition and mathematical problem-solving: preliminary research findings. Psychology of Mathematics Education, PME XI, J.Bergeron, N.Herscovics, C.Kieran (Eds). Montréal, 2, 222-228.

Locke E.A., Shaw K.N., Saari, L.M., Lathan G.P. (1981) Goal setting and task performance. 1969-1980. Psychological Bulletin, 90,125-152.

Marilier M.-C., Robert A., Tenaud I. (1987) Travail en petits groupes en terminale C. Cahier de Didactique des Mathématiques, 40, IREM Paris VII.

Pinard A. (1986) 'Prise de conscience' and taking charge of one's own congitive functioning. Human Development, 29,6,341-354.

Polya G. (1962-1964) On understanding learning and teaching problem-solving. T.I, II. New-York, Wiley and Sons, Inc.

Robert A., Rogalski J., Samurçay R. (1987) Enseigner des méthodes. Cahier de Didactique des Mathématiques, 38, IREM Paris VII.

Rogalski J. (1987) Analyse cognitive d'une méthode de raisonnement tactique et de son enseignement à des professionnels. Travail Humain, 4.

Rogalski M. (1987) Comment étudier la convergence d'une suite réelle. Question de méthode...Draft, Université Lille I.

Rogalski J., Samurçay R., Hoc J.-M. L'apprentissage de méthodes de programmation comme méthodes de résolution de problèmes. Draft. (Submitted to Travail Humain).

Schoenfeld A.H. (1985) Mathematical Problem-solving. Academic Press.

Schoenfeld A.H. (1987) Mathematics, Technology and higher-order Thinking. Draft, to appear in Technology in education in 2020: thinking about the not-distant future. R.S. Nickerson (Ed).

Vergnaud G. (1982) Cognitive and Developmental Psychology and Research in Mathematics Education: some theoretical and methodological issues. For the Learning of Mathematics, 3,2,31-41.

Vygotski (1962) Thought and language , M.I.T. Press.

Vivet M. (1987) Reasoned explanations, reasoning about reasoning and reasoning on the student. IFIP, TC3, Frascati.

STUDENT-SENSITIVE TEACHING AT THE TERTIARY LEVEL: A CASE STUDY[1]

Pat Rogers
York University

Abstract

Perceptions college students have of mathematics as a difficult and almost impossible subject can operate as a barrier preventing them from developing their full potential. This paper is about one department's success in changing those perceptions and creating a learning environment in which concern for the students' development overrides any concern about covering the curriculum. This approach succeeds in motivating students and encouraging them to high achievement in advanced level mathematics, at the same time fostering high self-esteem and confidence in their mathematical abilities, the ability to work independently and skill at proving theorems and reading mathematics. This study is the beginning of an attempt to describe conditions which favour the learning of more advanced and abstract concepts in mathematics.

"Proper curriculum is the heart of a mathematical sciences program, but there are many non-academic aspects that also must be considered." (CUPM, 1981) While this idea is a cliché at the elementary and secondary level, it has still had little impact at the post-secondary level. What research has been done into effective learning environments at the tertiary level has focussed on students who have previously had difficulty with mathematics (see for example Lochhead, 1983), rather than on mathematically able students. This paper is about one undergraduate department's success in balancing their concern for curriculum with a concern for developing each student to her fullest potential. In my study of this department I am attempting to describe conditions which favour the learning of more complicated and abstract concepts in mathematics.

In a 1981 report, the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America (MAA) cites examples of programs it has found to be successful in "attracting a

---

[1] This project is supported by a grant from the Social Sciences and Humanities Research Council of Canada under the Women and Work Thematic Program.

large number of students into a program that develops rigorous mathematical thinking and also offers a spectrum of (well taught) courses in pure and applied mathematics." The State University of New York (SUNY) at Potsdam College is one of those mentioned. According to an MAA survey (Albers at al., 1987) for the period 1980-85, while overall undergraduate enrolments in the United States remained relatively stable, there was an increase in the number of undergraduate mathematics degrees of 45%; the corresponding figure for Potsdam was 152%. Last year, just under one quarter of Potsdam's graduates had a major in mathematics and, of those who graduated with an overall average of at least 3.5 on a 4 point scale, over 40% were mathematics majors.

At most post-secondary institutions, complaints about the difficulty of attracting 'good' students, the low quality and inadequate preparation of the students they do have, and their inability to write a rigorous mathematical proof are common-place. People who make such complaints usually expect and find high drop-out rates in introductory courses and large numbers of students doing poorly on tests. One frequently also encounters the attitude amongst faculty that if too many students are successful in a course, then it cannot have been challenging enough. The main message of the CUPM Report is that rather than spend time complaining about students there is much to learn from the few departments cited where students are successful and quality and standards are maintained.

According to Poland (1987) the basis for the success of the Potsdam mathematics programme is that they "instill self-confidence and a sense of achievement through the creation of an open, caring environment." Students he talked to said they felt the faculty cared for each one of them and he observed that this was reflected in a high degree of confidence in their own mathematical ability. "The faculty win the students over to enjoy and do mathematics. It is simply the transforming power of love, love through encouragement, caring and the fostering of a supportive environment."

In October 1987, I began a study of the programme at Potsdam in order to identify and describe the programme's determining characteristics and to answer a number of questions raised by the Poland paper:

1. What precisely is the nature of the caring attitude the faculty at Potsdam display towards their students?

2. What specific teaching behaviours arise from this attitude towards students? (In his paper, Poland discounts teaching techniques as an explanation of their success.)

3. What are the specific aspects of their approach which are especially successful with their female students? (60.4% of the mathematics degrees awarded in 1983 went to women compared with 43.8% nationally. Degrees awarded to women in that year at Potsdam comprised 55% of the total number of degrees awarded compared with 51% nationally.)

4. What do the students think about the programme?

In this paper I shall confine my comments to summarising findings which relate to the first of these questions.

METHOD

This is an exploratory study using qualitative techniques to gather and analyze a variety of data. At the time of writing the first (data-gathering) stage of the investigation is complete and I have begun coding and analysing the data.

The data includes: all departmental printed material which is distributed to students; statistics relating to undergraduate enrolments, high school averages, and SAT scores for the last ten years; taped interviews with 40 students currently in the programme and with Dr Clarence Stephens, Chairperson of the department at Potsdam for eighteen years until his retirement last year. In addition, I have made extensive field notes of my observations including interviews with faculty members, counselling and admissions personnel; office consultations between faculty and students; and classroom sessions of almost all faculty members.

My final report will be presented to the faculty and students to check whether my findings match the experience of the participants. While most of what is reported here has been validated by one or two key informants, nevertheless, since this is a report of work in progress, the conclusions I have reached are tentative at this stage.

THE PARTICIPANTS

Potsdam College is a small undergraduate institution serving about 4000 students and is situated in the north east corner of New York State close to the Canadian/US border, a rural area known as the North Country. From its early beginnings in 1816 the college has been involved primarily in teacher education until it became the State University College of Arts and Science at Potsdam in 1962.

The mathematics department comprises 15 faculty members only one of whom is female and five of whom have joined the department within the last five years; all but one faculty member has a doctoral degree in mathematics. The teaching load varies between 9 and 11 hours each semester but in addition faculty may have one or more students doing independent study.

According to admissions personnel, the college draws from a wide area of New York State, attracting students primarily from lower middle class backgrounds, often from farming communities and small villages. Students are invariably the first in their family to attend college and with no tradition of post-secondary education to support them, poor self-concept and low self-esteem is often a problem.

MAJOR THEMES IN THE DATA AND DISCUSSION

The most striking feature of the programme at Potsdam is the learning environment. This has been created by establishing a balance between what the former chairperson would call a proper, rather than an excessive, concern for the curriculum and the standards of the department (Poland, 1987) and a concern for the development of their students. The faculty are highly sensitive to students believing that, "while the subject matter is important, the student is more so."

The predominant characteristic of this environment is its culture of success. Students at Potsdam are more concerned about whether they will do well enough to achieve high honours in a course rather than whether they will fail it. They expect to do well and they do. The faculty believe that it is their "job to teach the students they have, not the students they wish they had." Instead of complaining about the poor quality of their students, they work with the students at their level of understanding and develop them to their full potential. There is a strong belief in the students' ability to master difficult ideas in mathematics and this is communicated to the students who in turn come to believe in themselves.

What is the source of this belief in students? I think it owes its genesis to the experiences of the former chairperson, Dr. Stephens, when he taught in a black southern college and learned that "the perception students have about mathematics as an almost impossible subject has to be changed in order to teach them mathematics." Knowing this when he came to Potsdam, Stephens made it his primary focus to set about changing students' perceptions about the difficulty of higher level work in mathematics and whether they were capable of doing it.

A key strategy in Stephens' approach was to create role models. He did this by identifying students in their first year at the college who had demonstrated high promise in their course-work, and challenging them to do advanced level work in mathematics. At that time the department had no graduate programme and so he was faced with a dilemma: how could he motivate these students to "go very deeply into something when if they played around, after four years and they did less work" they would still get the same degree? For this reason, the department created the BA/MA double degree whereby students could get their undergraduate and graduate degrees concurrently in four years. This is an extremely demanding programme and over the years, less than 4% of their mathematics majors have graduated with the BA/MA degree, but its role in challenging the brighter students and providing examples to encourage and motivate ill-prepared students has been invaluable.

The spirit in which these role models have been used is also important. They are not held up as examples of excellence, as a means of rewarding the high achievers. Rather they are presented as examples of what can be achieved by any student who is prepared to put in the time and effort. The message received is: "Look at what these students have done. They're just like you. You can do it too." It is interesting to note that many of these early role models were women, one possible reason for the department's success in attracting female students.

Another way in which perceptions about the difficulty of upper level courses in mathematics are created is the tendency many departments have to give lower level courses to untenured faculty, part-time faculty, graduate students or faculty with no doctoral qualifications, and to reserve the upper level, 'more interesting courses', for senior faculty. Such a practice can convey to students the hidden message that upper level courses are so difficult that only the best, or the most experienced, or the few can teach them. Well may the student wonder whether, by implication, only the very brightest will be able to pass it. In a department which is sensitive to the perceptions of students, this is avoided by ensuring that all faculty teach across the curriculum. At Potsdam no one complains about teaching lower level courses because everyone gets the opportunity to teach upper level courses.

It has been interesting for me to observe how many of the attitudes towards students prevalent amongst the faculty at Potsdam are those attitudes considered, by proponents of effective parenting (see, for example, Dinkmeyer and McKay, 1976), to be crucial in building a child's self-confidence. The importance given by members of this department to building their students' confidence and self-esteem is central

Encouragement is an essential skill for building a students' confidence and self-esteem and the ways in which the faculty at Potsdam encourage their students are many and complex. One of these ways has already been discussed above: the deliberate creation of a rich tradition of role models and stories which place the student in a climate of success. Another way to encourage students is by recognising their efforts and accomplishments in much the same way that sports fans spur on their favourite team. Perhaps the most impressive way the Potsdam department does this is through their annual newsletter. Last year the newsletter was distributed to almost 2000 former students, high schools and graduates. In it were printed details of the new Clarence F. Stephens Mathematics Scholar award, the department's way of thanking and honouring its chairperson on his retirement. The award is to be given annually to "the non-graduating mathematics major who, by his or her achievement in mathematics, best personifies C. F. Stephens' vision of the mathematics student who is becoming all he/she is capable of being." The message is clear: You don't have to be *the* best, but you should strive to be *your* best. Competition is encouraged, but the competition is with oneself and the effort is recognised as well as the achievement.

Students are also encouraged by being challenged, but the challenges should be realistic. Instead of watering down the content and lowering standards as so often happens when faculty are concerned about giving their students success, the faculty at Potsdam believe that confidence comes from grappling with difficult ideas and concepts and being successful. And they are quite explicit about it, as one teacher told his students on the first day of class, "Frustration is a natural part of our game. My job is to keep you at the edge where you're frustrated enough to keep working but not too frustrated to quit." And they are prepared to provide the resources in terms of time and encouragement to support their students' efforts.

Other encouraging behaviours which I have observed are closely linked to their approach to teaching mathematics. This will be the subject of another paper so I will give only a brief sketch here. First and foremost the aim is to teach the student to think mathematically, to write a rigorous mathematical proof and to read a mathematics textbook. It is important not to race through the course in an attempt to cover a set syllabus - a student who has learned how to learn can cover the remaining course content by herself. Consequently, very few teachers at Potsdam adopt the traditional lecture format of teaching. Indeed some are quite vehement in their opposition to the method: "Suppose a person has a pile of sticks and they want to start a fire. They find two nice dry stones and they begin to rub them together. Then I walk in and pour a bucket of water over them. That's what a lecture is like!"

Instead a wide variety of teaching techniques are used which are relatively uncommon at this level of the educational system. Metaphors for describing this approach to teaching abound in the department but there is a common theme in all of them which agrees with what I have also observed in the classroom. There is the acknowledgement that becoming a mathematician, like becoming an athlete, takes time, practice and lots of encouragement and support. The teacher at Potsdam is a coach.

To summarize, some of the techniques I have observed are: active student participation and group work in class and outside of class; 'coming to the blackboard'; learning by helping others informally and more formally in the student run Math Lab; observing seatwork; an approach to grading tests and homework that construes them as articles of learning rather than measures of ability; a flexible grading scheme which allows for the student who blossoms late in the course; explicit teaching in how to read a mathematics text with understanding; and most importantly constructivist approaches to developing the subject matter.

CONCLUSION

The learning environment at Potsdam has been created by challenging perceptions students have about the difficult nature of mathematics which inhibit their ability to succeed at it. In creating this environment, faculty have been motivated by a concern or caring for students which is directed towards helping them become the best they are capable of being. The faculty believe that developing a student requires time, encouragement and challenge and that the best way to do this, as Stephens would say, is to 'go fast slowly'. In other words teachers who are sensitive to the needs and level of understanding of their students will sequence instruction at a pace at which students can learn. This is the essence of a student-sensitive approach to teaching because concern for the individual student's development overrides any concerns about covering the curriculum.

One consequence of this student-sensitive approach is that, as news about the department's success with students has reached the high schools, they now attract better prepared students. Presently one of the most selective of the SUNY colleges, Potsdam attributes part of its success in attracting good students to the excellence of the mathematics department. Over one quarter of the incoming freshmen at Potsdam were in the top ten of their high school classes and the college's freshmen score the highest mathematics SAT scores in the whole SUNY system.

Given the large numbers of majors they teach and the nature of their approach to teaching mathematics, it is natural to ask what compromises in curriculum have been made. No compromise in standards has been made, in fact the opposite is true. But certain economies have been made in order to keep class size down to the level (40) the department insists upon. The mathematics major is a minimal degree requiring 30 credit hours (10 one-semester courses) in a very traditional, pure mathematics sequence with a limited range of options.

The experience of graduates of the program who have gone on to jobs with companies like IBM, Kodak, and Hewlett-Packard suggests that students leave Potsdam with excellent work skills: the ability to think independently, read and write technical reports, work cooperatively with other people, present and defend their work, and also offer criticism to others without annihilating them. Students who have gone on to graduate school, at places like Cornell, Illinois, Michigan and Wisconsin, report that while their mathematical preparation may not be as broad perhaps as other students, their learning skills enable them to bridge any gaps for themselves and that they are well prepared for independent work at the graduate level.

Graduates of Potsdam College are very loyal to the mathematics department. Many of them have mentoring relationships with a faculty member and keep in touch for years after they leave. Some return to speak at ΠME (their honorary mathematics society) functions, providing role models for current students and living proof of the value of a pure mathematics education that taught them more than a collection mathematical facts.

Bibliography

Albers, D. J., Anderson, R. D. and Loftsgaarden, D. O. (1987). "Undergraduate Programs in the Mathematical and Computer Sciences: the 1985-86 Survey." MAA Notes, Number 7.

CUPM. Report (1981). "Recommendations for a General Mathematical Sciences Program." Mathematical Association of America.

Dinkmeyer, Don and McKay, Gary. (1976) "Systematic Training for Effective Parenting." American Guidance Service, Inc., Circle Pines, Minn.

Lochhead, Jack. (1983) "Constructivist Approaches to Teaching Mathematics and Science at the College Level". Proceedings of the Fifth Annual Meeting of PME-NA, Montreal, 2, 74-80.

Poland, John. "A Modern Fairy Tale?" The American Mathematical Monthly, 94 (3), 291-295.

STRATEGY CHOICE IN SOLVING ADDITIONS: MEMORY OR UNDERSTANDING OF NUMERICAL RELATIONS?*

Analúcia Dias Schliemann
Universidade Federal de Pernambuco, Recife, Brasil

Siegler's strategy choice model was tested for additions of two addends ranging from 1 to 29, on a sample of 20 Brazilian street vendors, very skilled in mental computation. The model proved to be adequate for addends no larger than 10. For larger addends, properties of the decimal system, more than memorization, seems to better predict the strategy used.

Data on everyday mathematics (Carraher, Carraher & Schliemann, 1985, 1987) have shown that, when computing the results of arithmetical operations, most of the time children use oral procedures. One of the most common of these procedures is the decomposition strategy. When using decomposition to calculate, for example, the result of 95 + 57, one might add 90 (from the first addend) and 50 (from the second), obtaining 140, which is than added to 12, the result of 7 + 5, yielding 152. Such a strategy, as already demonstrated by Carraher & Schliemann (1988, in press) reveals a clear understanding of the decimal system and of the properties of the additive composition of numbers.

How these strategies develop and how they relate to memorization of addition facts is still unknown. Experiments by Siegler & Robinson (1982) and by Siegler & Schrager (1985) analyzed the strategies used by 4- and 5-year-olds to solve additions of two addends with values from 1 to 5 or from 1 to 11, with sums no larger than 12. Siegler (1986) proposed that the choice of a strategy among others would be determined by the strength of the associations between the pairs of numbers to be added. This association was determined in a separate experiment where 4-

---

*This research was supported by a grant from CNPq, Brazil. I am indebted to R. Siegler, for discussions that originate the study, to D.Carraher for his comments on a first version of this paper and to Amy, Carlos, Enildo, Ivo, Leila, Patricia and Solange for data collection and analysis.

and 5-year-olds were asked to say what they thought were the right answer for the addition of each pair of numbers, without putting up fingers or counting. The amount of correct and incorrect answers thus obtained for each pair of numbers was used to determine the strength of the associations between these numbers. This distribution of associations model, tested in different types of tasks, proved to be adequate to predict the strategies used in those simple additions. The strategies found ranged from overt behavior strategies (finger counting, finger display with no apparent counting, verbal counting) to memory retrieval where no overt behavior was observed. Solution times, degree of overt behavior displayed and eficiency in solving the additions, were all highly correlated (around .90) with the degree of association found between the pairs of numbers involved in the additions.

What would happen, however, with additions involving larger numbers, allowing use of other more soffisticated procedures such as decomposition strategies? Would the same strategy choice model apply? Siegler (1986) proposes that his model would hold for additions, subtractions and multiplications. However, Hope & Sherrill (1987), in their study on the characteristics of skilled and unskilled mental calculators have shown that performance on mental multiplication of large numbers had a low positive correlation with general multiplication fact recall.

It has been shown that oral addition of numbers frequently involve decomposition strategies. These strategies could either be determined by the addition facts recalled by the subject or by a general understanding of the decimal system. In the first case, as predicted by Siegler's model, the decomposition strategy should be used for numbers not recalled by the subject and the kind of decomposition used should be related to the addition facts they know. If, however, understanding of the relations involved in the decimal system is a more prominent factor than memorization of addition facts, even for pairs memorized by a subject, decomposition strategies would often be used and the way numbers are decomposed would not be related to the memorized addition facts but most often to the decimal system properties.

This study aims to find out the relative importance of memorization versus understanding of the decimal system in the choice of different strategies for solving additions.

METHOD

Subjects: Twenty 9- to 13-year-old Brazilian children, who worked as street vendors participated in the study. Their school experience was irregular and they attended, or had attended, at most, the 3rd grade. At work, when selling candies, lollipops, ice-cream, fruits or sandwichs, they were used to mentally compute the results of additions, subtractions and multiplications.

Material and Procedure: Subjects were asked to orally solve a series of 216 two-addend additions. In the first phase of the study they were instructed to answer each of the 216 pairs of numbers to be added, as quickly as possible and, when two seconds were elapsed, if no answer was given, another pair was presented. In the second phase they were asked to orally solve, in a different order, the same 216 problems, using whatever methods they want and explaining how they reached each result. Of the 216 additions, 45 involved the addition of two numbers from 2 to 10 with the larger addend preceding the smaller one; 171 involved the addition of a number between 21 and 29 with another in the interval of 2 to 20. These 216 additions could be classified according to the numbers involved, into five groups, as shown in Table 1.

Table 1
Classification of the 216 additions presented to the subjects in the first and the second phases

Problem type	Values of Addends First Addend	 Second Addend	N of Problems
Group 1	1 to 9	1 to 9	36
Group 2	10	1 to 10	9
Group 3	21 to 29	1 to 9	72
Group 4	21 to 29	10 or 20	18
Group 5	21 to 29	11 to 19	81

RESULTS

The percentage of memorized pairs, for each type of addition--That is, those correctly answered in less than two seconds--is shown in Table 2. Memorization was at its highest for group 1 problems (10 plus a number from 1 to 10) where 86.1% of problems were solved. Group 1 (1 to 9 plus 1 to 9) and group 4 (21 to 29 plus 10 or 20) followed with 59.6% and 46.7% of correct answers, respectively. The most difficult additions were those in groups 3 (21 to 29 plus 1 to 9) and 5 (21 to 29 plus 11 to 19) which were solved in only 31.9% and 9.0% of the cases, respectively.

Performance in the second phase was nearly errorless: only 22 errors were found among the total of 4320 problems presented to the 20 subjects. The prefered strategy to solve type 1 and type 2 additions was memory retrieval. For types 3, 4 and 5, decomposition was the strategy most oftenly used. Counting strategies appeared in a few problems, either in isolation or combined with decomposition. This general analysis seems to show that the data obtained support Siegler's model: for the additions solved in the first phase, more memory retrieval was found in the second; for those not solved, other strategies were chosen. However, if a more specific analysis is performed a different picture may appear.

Table 2
Percentage of problems solved in the first phase and percentage of problems solved through memory retrieval, decomposition and counting strategies in the second phase.

Problem		Solved in	Strategy in 2nd phase			
Type	N	1st phase	Mem.	Dec.	Coun.	Mix.
Group 1	36	59.6	64.6	20.7	14.2	.4
Group 2	9	86.1	92.7	2.8	4.5	.0
Group 3	72	31.9	23.5	56.8	17.2	2.4
Group 4	18	46.7	38.7	52.9	7.5	.9
Group 5	81	9.0	6.9	74.7	10.2	8.2

Table 3 shows, for problems that were solved in the first phase, the percentage of those where the answer in the second phase was given through each one of the strategies.

The same data are also shown for problems not solved in the first phase. Siegler's model leads to the prediction that, for pairs of numbers with a strong association--in this study those solved in the first phase--memory retrieval should be the chosen strategy in the second phase. For those pairs not solved in the first phase, other strategies such as decomposition or counting would be chosen more frequently in the second phase. Inspection of Table 3 reveals that this prediction applies only to type 1 problems. For these, as predicted by the model, use of memory retrieval was much more common for the problems solved in the first phase than for the unsolved ones. Type 2 problems show a very high percentage of use of memory retrieval for solved problems, but this also happens for the ones left unsolved. For types 3, 4, and 5, regardless of the results in the first phase, there was a clear preference for strategies other than memory retrieval, mainly the decomposition strategy.

Table 3
Mean and percentage of solved (S) and unsolved (U) problems in the first phase and percentage of use of memory retrieval, decomposition, counting, and mixed strategies in the second phase, for corresponding problems.

Problem type			1st phase		Strategies in 2nd phase			
			Mean	%	Mem.	Dec.	Coun.	Mix.
Group 1	36	S	21.4	59.6	78.8	14.0	7.2	0
		U	14.0	38.9	42.8	31.1	25.0	1.1
Group 2	9	S	7.7	86.1	95.5	2.0	2.5	0
		U	1.2	13.3	75.0	8.3	16.7	0
Group 3	72	S	23.0	31.9	26.2	69.4	4.4	.6
		U	48.7	67.6	22.3	51.3	23.2	3.3
Group 4	18	S	8.4	46.7	33.3	57.7	8.3	.6
		U	8.9	49.4	43.8	48.3	6.7	1.1
Group 5	81	S	7.3	9.0	13.0	83.6	3.4	0
		U	73.5	90.7	6.3	73.8	10.9	9.0

* Total means and percentages in the first phase differ from total number of problems because a few problems were not presented in the first phase.

The model also leads to the prediction that the number of problems solved in the first phase should be highly correlated with the number of these same problems solved through decomposition in the second phase. Also, the number of unsolved problems should be highly correlated with the number of these same problems solved through strategies other than memory retrieval. Table 4 shows, for each addition type, the correlation coefficients obtained for this analysis. For types 1 and 2 problems, the simpler ones, involving addends no larger than 10, use of memory retrieval was in fact highly correlated with number of solved problems in the first phase. However, this correlation was also very high and significant for the unsolved problems. For types 3, 4, and 5, problems with at least one addend larger than 20, the correlations go clearly against the model: use of decomposition was always highly correlated with number of solved but not with number of unsolved problems. Only counting strategies, for all problem types, shows significant, although not very high correlations with number of unsolved problems, as predicted by the model.

Table 4
Kendall's tau correlation coefficients for number of solved (S) and unsolved problems (U) in the first phase with number of these problems solved through each strategy in the second

Problem		First phase		Strategy in second phase			
type	N	Mean S and U		Mem.	Dec.	Coun.	Mix.
Group 1	36	S	21.4	.60***	.25	-.10	-
		U	14.0	.57***	.27	.39**	.10
Group 2	9	S	7.7	.73***	-.03	.08	-
		U	1.2	.79***	.15	.19*	-
Group 3	72	S	23.0	.16	.75***	-0.3	.02
		U	48.7	.44**	.04	.27*	.01
Group 4	18	S	8.4	.23	.49**	-.19	.00
		U	8.9	.23	.20	.25*	.34*
Group 5	81	S	7.3	.29*	.74***	.09	-
		U	73.5	.08	.14	.24*	.18

*** $p < .001$, ** $p < .01$, * $p < .05$.

For types 3, 4, and 5 problems the decomposition strategy consisted in most of the cases in separating, for each addend, the tens from the units. The tens and the units are then added together separately and the two results added at the end of the procedure. Variations within this general approach were related to the order the units to be added were mentioned (nearly always the larger one was mentioned first), and the order the tens and units were taken (there was a general tendency to add the tens first). When the sum of the units was larger than 10, and was obtained before the sum of the tens, a second step in the procedure could appear which consisted in decomposing the result of the sum of the units into ten plus units, add the ten to the original tens, joining the units that were left at the end.

For problem types 1, 3, and 5, another kind of decomposition, used with different frequencies by 19 of the 20 subjects, appeared in a total of 746 problems. This consisted in adding, to one of the addends or to its units, part of the units of the other addend so that 10, 5, or a multiple of 10 or 5 was obtained. In types 3 and 5, when 10 or 5 was obtained, it was joined to the original tens, if there were any, and the part of the units was aggregate at the end to the round number obtained (multiple of 10) or to the multiple of 5, if this was the case. In most of the cases where 10 or a multiple of 10 was searched, the units of one of the addends were of value 8 or 9. Examples of this strategy are the following answers:

> 9 + 3? "12, I added 1 to 9, there was 2 left; I added 2 to 10."
>
> 28 + 19?"28 plus 19, let me see (pause) 28 plus 19 (pause) 40 (pause) 47. This one I took 10 from 19 and put it on 28. Then I took 2 from 9, and I had 40. There was 7 left, it makes 47."

Among the problems solved through this sort of composition, more than one third involved the additions of o units that were solved in the first phase (in type 1 oblems). Correlation coeficients for number of type 1 ditions solved in the first phase with use of this second rt of decomposition strategy in each problem type tended be negative but were all very low and non-significant.

CONCLUSIONS

The choice of strategies to solve addition problems, although influenced by memorization of addition facts, seems to be also strongly determined by the understanding of the characteristics of the decimal system, by the situation where the problems are solved, and by the kind of numbers to be added. Thus, Siegler's strategy choice model, although adequate to explain the choice of simpler strategies to solve addition of small numbers by young children, who often rely on counting strategies, does not seem to fit the case of more complicate additions solved by skilled mental calculators, who use different sorts of decomposition strategies. Of course one can always argue that the decomposition strategies themselves are determined by the stronger association that exists for 10 and multiples of 10 with numbers smaller than 10. But this association only holds if an understanding of the decimal system as a generative system pre-exists. For numbers larger than 10, when the child understands the relations involved in the system, the role of memory skills is reduced. Understanding the decimal system allows the child to find out, whenever needed, the results of additions, making school training or memorization of addition facts irrelevant.

REFERENCES

Carraher, T.N., Carraher, D.W. & Schliemann, A.D. (1985). Mathematics in the streets and in schools. British Journal of Developmental Psychology, 3, 21-29.

Carraher, T.N., Carraher, D.W. & Schliemann, A.D. (1987) Written and oral mathematics. Journal for Research on Mathematics Education, 18(2), 83-97.

Carraher, T.N. & Schliemann, A.D. (1988) Culture, arithmetic and mathematical models. Cultural Dynamics, in press.

Hope, J.A. & Sherril, J.M. (1987). Characteristics of unskilled and skilled mental calculators. Journal for Research in Mathematics Education, 18(2), 98-111.

Siegler, R.S. (1986). Unities across domains in children's strategy choices. In M. Perlmutter (Ed.), Minnesota symposium on child development (vol. 19) Hillsdale, N.J. Erlbaum.

Siegler, R.S. & Robinson, M. (1982) The development of numerical understandings. In H. Reese and L.P. Lipsitt (Orgs.), Advances in child development and behavior. New York, Academic Press.

Siegler, R.S. & Schrager, J. (1984). A model of strategy choice. In C. Sophian (Org.), Origins of cognitive skills. Hillsdale, NJ: Lawrence Erlbaum Associates.

REPRESENTATIONS OF FUNCTIONS AND ANALOGIES

Baruch Schwarz & Maxim Bruckheimer
Weizmann Institute of Science, Israel

A major difficulty in the learning of functions is the transfer of knowledge and methods between representations. The computerized environment T.R.M. was created to alleviate this difficulty. A series of studies on learning processes with T.R.M. was undertaken. This paper reports on an investigation of students' use of analogies in transferring knowledge between representations.

REPRESENTATIONS AND ANALOGIES

Although the concept of function and its subconcepts are not theoretically linked to a particular representation, the curriculum of necessity translates these concepts into several representations. The preimage-image link, for example, may be represented algebraically in the form $y_0=f(x_0)$, graphically by a point, or by a pair of data in a table. Similarly, other notions have to be based on one or more representations. Typically, three or four representations are used in the initial study of functions. The passage between these representations is difficult (see e.g. Markovits et al., 1986). The properties of a function are often understood in their representational context only and no abstraction of these properties is made by beginning students (nor, often, by more advanced ones).

Such a tendency to compartmentalize knowledge has been noticed in several domains. Schoenfeld (1986), in geometry, showed how students who acquired knowledge in one context kept it separate from knowledge acquired in other contexts. Kaput (1982) obtained similar results in algebra. Greeno (1983), on the other hand, indicated how analogies can facilitate the construction of relationships between units of knowledge:

> "If the domains are represented by entities that have relations that are similar, the analogy may be found easily, but if the representation of either domain lacks these entities, the analogy may be impossible to find. Consequently, an analogy can be used in facilitating the acquisition of representational knowledge in a domain." (p.228)

The representational domains of a function are composed of quite different objects and the methods which are used in each representation are quite different from each other. For example, the solution of an equation of the form $f(x)=a$ can be obtained by algebraic methods (extracting roots, simplifications ...) or by constructing and reading the graph of the function. The need to establish these representational domains and the relationships between them led us to construct a computerized environment, the Triple Representational Model (T.R.M.), whose principal characteristics are:

* T.R.M. facilitates transfer of function concepts between three representations: algebraic, graphical and tabular. The technical tasks are executed automatically; the student has to organize and to relate results linked with one representation in order to use them in others.
* Work within each representation is operational, i.e organized in terms of operations that the student has to carry out.
* T.R.M. is the computerized core of a complete Grade 9 function curriculum based on problem solving and exposes the student to a great variety of functions.
* The construction of T.R.M. is intended to provide a good ontology of domains which facilitates analogies between representations. Therefore, operations available in the three representations were chosen to be conceptual entities whose utilization is similar. Their detailed description will be given in the following.

BRIEF DESCRIPTION OF T.R.M.

Three typical operations will be described to convey the character of T.R.M.: "Search" (algebraic), "Compute" (algebraic) and "Draw" (graphical).

"Search" enables the user to solve (in)equalities involving the function f(x) under consideration. The structure of this operation is shown in Fig.1. Using this operation the student can search, for example, for the zeros of a function, for the subdomains in which the function is increasing (see Fig.2), solve inequalities (e.g replace f(x+0.01) by 0 in Fig.2 etc.. The "Search" operation changes the conventional aspect of the algebraic representation based on intensive computation to extensive computation which is performed by the software.

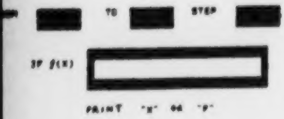
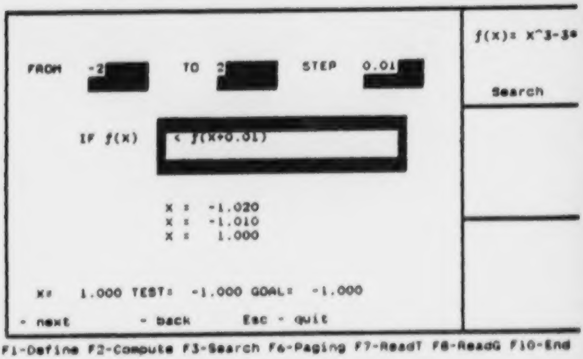


Fig1. The Search operation



```
f(x)= x^3-3*

FROM  -2        TO  2        STEP  0.01

IF f(x)   < f(x+0.01)

          X :  -1.020
          X :  -1.010
          X :   1.000

X=  1.000 TEST=  -1.000 GOAL=  -1.000
```

Search
- next
- back
Esc - quit
F1-Define F2-Compute F3-Search F6-Paging F7-ReadT F8-ReadG F10-End

Fig.2. Search for subdomain
in which f(x) is increasing

"Compute" enables the student to compute automatically the value of a function for any desired element of the domain.

"Draw" enables the student to draw the graph of a function, to zoom on subdomains or to stretch the graph in one direction. This operation not only removes the technical fatigue but adds a dynamic aspect to the graphical representation.

In addition to the fact that T.R.M. enables the user to move or to read information between representations, its operations diminish the conceptual distances between the representations by stressing operational parallels. Two general procedures in the T.R.M. are directly parallel in the graphical and algebraic representation:

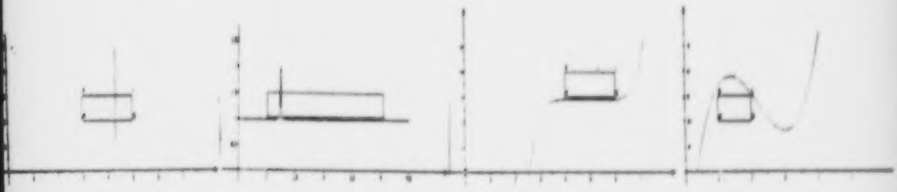
1) Convergence, by which the student "homes in" on the desired result. It is realized in the graphical representation by a well judged sequence of zoomings and in the algebraic representation by intelligent use of the "Search" operation.
2) Accuracy of the required result, which uses the same operations as convergence but is supported by other strategic considerations.

We focused on these two procedures because the operations of T.R.M. facilitate analogies between the graphical and the algebraic representation with these procedures.

AN EXPERIMENT WITH T.R.M.

The T.R.M. curriculum has been taught experimentally in two 9-th grade classes in junior high schools. In this paper we report on one aspect of this experiment, the procedural analogy between the graphical and the algebraic representation. The treatment of the two classes differed only in the order of the learning process. One class (C1) was introduced to functions through the graphical representation, whereas the other class (C2) was introduced via the algebraic representation. At this stage C1 was given the computer-based task CIN1 (CIN= computerized interview) and C2 was given CIN2 (see below). In the second stage each group was introduced to the other representation and given the other task. Some students performed the tasks in a classroom setting and some were interviewed individually. CIN1 was essentially graphical in form and CIN2 algebraic:

CIN1: The computer chooses an undisclosed function f and displays a rectangle on the screen. The student is asked to decide by interrogating the computer whether the graph of the function passes through the rectangle. The hidden graphs took one of the following four forms:



and the student could only use the Compute operation to solve the task.

CIN2: "Find an x which satisfies f(x)=a to an accuracy of 0.001." The student could use the Compute and Search operations.

Collection of data: A program was written to record student behaviour on the tasks. For example, in one of the CIN1 tasks the given rectangle was as shown in Fig.3 and the student's trials (using the Compute operation) are shown numbered. The student concluded that the graph passes through the rectangle, which, in this case, is false (see Fig.4).

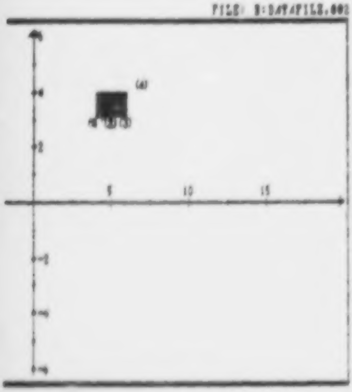


Fig.3. Student trials in CIN 1

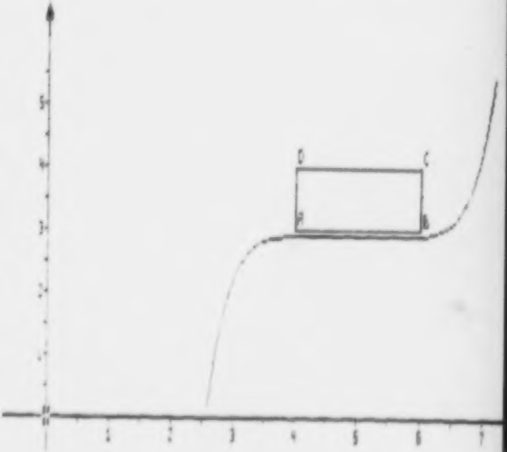
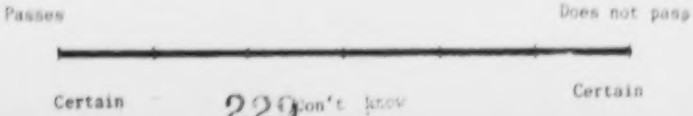


Fig.4. Rectangle and hidden graph in CIN1

The order in which the student calculates the points tells us about the student's use and understanding of the convergence procedure. We also asked students to express their confidence in their conclusion on the following diagram and this tells us something about their understanding of the accuracy procedure.



An example of a CIN2 task was to solve $x^3-3x+4=5$, for which the correct answer is $x=1.879$. Fig.5 shows the various trials of one student, who used the Compute operation for the first three trials and then the Search operation for the next two and finally returned to the compute for the final three. Since she wrote the condition for Search operation in the form "IF $f(x)=5$" her search was "fruitless". However, either on the basis of the $f(x)$ values in the Search or the computed values in the first three trials, she completed the task successfully.

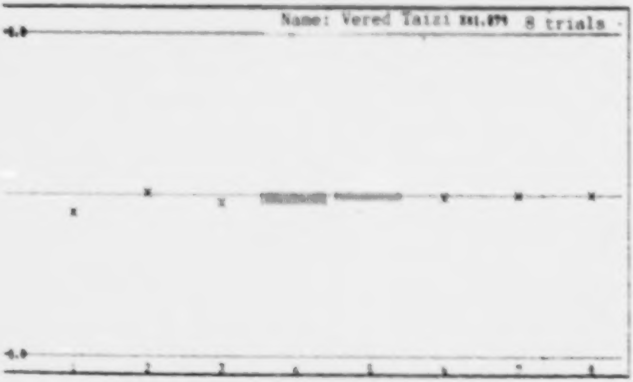


Fig.5. Student trials in CIN2

There is clear indication that this student appreciates the convergence procedure but has trouble with the accuracy as shown by her Search operations. To resolve her difficulties she returns to Compute. Other students tackled the accuracy problem by changing the Search condition to "IF $f(x) > 5$" together with an appropriate choice of step-length.

RESULTS AND FINDINGS.

The design of CIN1 and CIN2 enabled us to check two different questions:

Analysis of characteristic student behaviour in CIN1 and CIN2

For CIN1, a three level categorization scheme was found to be appropriate:

1) The student computes irrelevant images outside the bounds of the rectangle. Decisions are based on linear interpolation only and confidence is low.
2) The student computes relevant images and his search is systematic. The computations are managed by linear interpolation and confidence is high.
3) The student manages the computations by interpolation and continuity, confidence is high, and well-founded in discussion.

For CIN2 a similar categorisation was found to be appropriate:

1) The student works randomly with the Compute operation until a direction is found for the search. Efficient use of the Search operation operation is not made; the number of trials is large.
2) The student's analysis process converges almost from the beginning and intelligent use of previous computations is made. Not much use of the Search operation is made and then always with equalities rather than inequalities.
3) The student integrates Compute and Search operations in an efficient converging solution.

From the behaviours observed with CIN1 and CIN2, we arrived at the following sketch of general cognitive levels of functional thinking.

1) The numerical level: The functional link between preimages and images is not well understood. The search for the result is not systematic.
2) The functional reasoning level: The functional link between preimages and images is understood. The search for results is systematic but does not use a logical sequence of computations.
3) The dynamic functional reasoning level: The student understands the richness of the concept of function and can search for a result by an efficient converging sequence of computations.

Comparison of the achievements on CIN1 and CIN2 within/between C1 and C2

The analysis of the data showed that C1 (which started with the graphical representation) had results in CIN1 similar to those of C2, and much better results in CIN2. This would seem to indicate that learning the graphical representation first leads to a higher level of functional reasoning. We also found that accuracy and convergence procedures transfer from the graphical to the algebraic representation but not in the opposite direction. However, as Gick and Holyoak (1983) noticed, if two prior analogs are given, students can derive an underlying principle as an incidental product of describing the similarities of the analogs. Consequently, a theoretically based function curriculum which integrates the various representations at an early stage, may well have advantages over either system used in this experiment.

REFERENCES:

Gick, M.L. & Holyoak, K.J. (1983). Analogical transfer. Cognitive Psychology, 15, 1-38.

Greeno, J.G.(1983). Conceptual entities, In D. Gentner & A. Stevens, Mental models. 227-252

Kaput, J., (1982). Intuitive attempts at algebraic representation of quantitative relationships. Paper presented at the annual meeting of the American Educational Research Association, New York.

Markovits, Z., Eylon, B., & Bruckheimer M.(1986). Functions today and yesterday. For the Learning of Mathematics. 18-24.

Schoenfeld, A. (1986). On having and using geometric knowledge. In Conceptual and Procedural Knowledge: The Case of Mathematics. J. Hiebert(Ed.). 225-264.

OPERATIONAL VS. STRUCTURAL METHOD OF TEACHING MATHEMATICS - CASE STUDY

Anna Sfard

The Hebrew University of Jerusalem

The "operational" method of teaching mathematics was first proposed in [4], on the grounds of certain theoretical claims and experimental findings dealing with the learning of advanced mathematical concepts. In the present study the method is applied to mathematical induction. The new approach is compared, both theoretically and experimentally, to the conventional ("structural") way of teaching the subject.

INTRODUCTION

The experimental study which will be reported in this paper is a continuation of our extensive research on the role of algorithms in the acquisition of mathematical concepts. The theoretical framework and the initial stages of this research have been presented in [4]. Here we shall describe our first attempt at examining the didactical implications of the former study.

In [4] we suggested that abstract mathematical notions can be conceived in two fundamentally different ways: either structurally or operationally. People who think structurally refer to a formally defined entity as if it were a real object, existing outside the human mind. Those who conceive it operationally, speak about a kind of process rather than about a static construct. Both approaches play an important role in all kinds of mathematical activities. The process of solving problems consists in an intricate interplay between the structural and operational versions of the appropriate mathematical ideas. Since computational procedures are more "tangible" then abstract mathematical constructs, it seems plausible that formation of an operational conception is, in many cases, the first step in the acquisition of a new notion. Two experimental studies, presented in [4], provided some initial evidence for this conjecture.

The structural approach predominates in the most developed branches of contemporary mathematics. Accordingly, structural definitions and representations are taught at universities and in schools, while very little attention is given to the processes underlying the mathematical concepts. The appropriate algorithms are never explicitly formulated: it is tacitly assumed that by the help of structural definitions the processes become self-evident, and that only a little training is needed to ensure, that they will be correctly executed whenever necessary. In the light of our former claims, more direct treatment of algorithms can greatly improve the learning. If the operational conception is indeed the necessary first step in an accquisition of a new mathematical idea, we can probably make the learning more effective by communicating with the student in the suitable "operational" language, and by fostering the pupil's understanding of processes before translating the operational descriptions into structural definitions. All this can be done by incorporating computer programming into mathematics courses. While writing the programs the student would get a profound insight into the algorithms underlying a mathematical concept. This should deepen the understanding of the concepts themselves and create a sound basis for the transition from operational to structural conception. [Formation of the structural conception of basic mathematical ideas seems to be essential for further learning - for acquisition of more advanced concepts. If so, the structural conception should be promoted in behalf of those pupils who are able and willing to continue their mathematical education after matriculation.]

In the experimental study, which will now be described in detail, we tried to compare the effectivness of the "structural" (conventional) and the "operational" (the one proposed here) methods of teaching. Mathematical induction has been chosen as a perfect subject for this kind of investigation. Firstly, the topic can be easily presented in two ways, both struc- turally and operationally. Secondly, while being one of the most important mathematical ideas taught in (Israeli) secondary school, it is also considered to be particularly difficult for the learner. As such, it has already inspired quite considerable amount of both theoretical and experimental educational studies ([1], [2], [3], [5]).

STRUCTURAL APPROACH TO MATHEMATICAL INDUCTION

The way mathematical induction is taught in Israeli senior secondary-schools may be regarded a typical implementation of the structural method. According to the curriculum, 20 teaching hours should be devoted to the subject in eleventh or twelfth grade. Let us describe now the main stages of the learning, and at the same time indicate and analyse the difficulties which may be encountered by the learner at each of them.

1. Recursion. To begin with, the student is presented with the idea

of sequence. It is assumed that the pupil is already well acquainted with the concept of function (in its structural version!), so the sequence can be considered nothing more than a particular case of the familiar mathematical construct. The recursive representation of a sequence (see Box 1(a)), however, is a new idea, which is explained by help of appropriate examples and exercises. Since the sequence is presented as a static entity (composed of infinitely many parts), the rule of recursion can only be perceived as a constant relation between its adjacent components.

Here, a serious difficulty may stem from a quite unusual role played by the variable n. To find a rule of recursion for a given sequence (such as $f(n)=n^2$), the student has to begin with the substitution of n+1 instead of n into the formula which defines the sequence. Until now, the variables such as n were used only in their structural, static sense: any letter denoted an unknown magnitude, which was assumed to remain constant throughout the entire process of solving a given problem. Now, for the first time, the students must cope with an additional, dynamic meaning of a variable. While substituting n+1 instead of n, they have to be aware that the letter n serves both as a "given number" (n+1 is its successor), and as a "cell" for storing changing magnitudes. This double role of n may be quite confusing for unprepared learners. The bewildered student would make such classic mistakes like adding 1 to f(n) while looking for f(n+1).

2. The principle of mathematical induction is introduced in its formal axiomatic version, as presented in Box 1(b).

It seems pretty obvious that the fully developed structural conception of the notion of infinite set is indispensable for understanding the underlying idea of equality between K and N. The way the two sets are compared may be an inexhaustible source of additional difficulties. Everyday classroom experience clearly shows that many students cannot get along with the statements of the form $\forall n\, [P(n) \Rightarrow P(n+1)]$, which constitute the very heart of an inductive proof.

To get a deeper insight into the problem, we asked 16 students who had just finished the regular twenty-hour course on induction to describe the main stages of inductive proof (see Test 2 in Box 2). Only four answers could be regarded as correct. The remaining twelve responses clearly indicated some serious problems with quantification. More often than not, the statement $\forall n\, [P(n) \Rightarrow P(n+1)]$ was transformed into $[\forall n\, P(n)] \Rightarrow P(n+1)$. Indeed, seven students wrote: "We have to show that if f(n)=g(n) for every n, then f(n+1)=g(n+1)" or "Let us assume that the functions are equal, and then show that f(n+1)=g(n+1)".

Box 1: Structural vs.operational approach to math. induction

	Structural	Operational

Representation of recursively defined sequence	(a)
	$\begin{cases} f(0)=0 \\ f(n+1)=f(n)+2n+1 \end{cases}$
	(recursion rule)
		(c) y:=0;
		n:=0;
		while $n<n_0$
		begin
		y:=y+2n+1; (recursion rule)
		n:=n+1;
		end;

The principle of induction	(b) IF $K\subset N$ and if a. $0\in K$ b. for every $n\in N$ if $n\in K$ then $n+1\in K$ THEN $K=N$	(d) IF two sequences, f and g, a. have the same initial value b. can be computed by the same recursion rule THEN $f(n)=g(n)$ for every n.

The same misplacement of quantificators might be responsible for another common answer (5 cases): "We have to prove that if $f(0)=g(0)$, then $f(n+1)=g(n+1)$". It seems quite likely that the students skipped the inductive assumption simply because they felt that the "premise" they were going to use was identical with the proposition which had to be proved.

This kind of mistake can be easily explained on the grounds of our former study, devoted to the notion of function [4]. According to our findings - and contrary to the expectations of the designers of secondary-school curricula - the majority of pupils do not conceive function as "an aggregate of (infinitely many) ordered pairs". Rather, they identify it with a certain computational formula. For these students, two functions are equal only if one of the appropriate formulas can be obtained from the other by certain algebraic manipulation. If so, the quantificators have no significance whatsoever, and the equation "$f(n)=g(n)$" is equivalent to the statement "f and g are equal"

Finally let me remark that the students who do manage to put the quantificators in the right places, may still have some problem with the general logical structure of the axiom. If an induction is a method for proving the propositions beginning with the words "For every $n\in N$...", the question can rightly be asked, why not use this very method inside the inductive proof, while dealing with the statement $\forall n\,[\,P(n) \Rightarrow P(n+1)\,]$. Since it seems that the method should be used (recursively!) over and over again, the student may feel entangled into a vicious circle.

3. Proving by induction. The principle of induction is applied in a series of proofs dealing with various properties of numerical sequences. The problems 2 and 3 presented in Box 2 (Test 3) are two typical examples of exercises apearing in the conventional textbooks.

OPERATIONAL APPROACH TO MATHEMATICAL INDUCTION

For our experiment, new teaching material on induction was prepared. This time the subject was presented in an operational manner. While describing the main stages of learning we shall argue now, that within our special approach most of the previously described difficulties can be either easily overcome or avoided altogether.

1. Recursion. According to our program, at the first stage of learning the pupils get acquainted with many kinds of recursive calculations. On the grounds of our previous experimental findings ([4]) we assume, that the majority of students conceive function (sequence) as a computational process, rather than as a static construct. Accordingly, a rule of recursion is presented as a prescription for some special kind of computation. The student's task is not only to understand and to execute recursive operations (represented by the suitable algebraic expressions), but also to formulate iterative algorithms for recursively defined functions in a simple programming language (see Box 1(c)). This additional, operational representation is an effective tool for dealing with a new, dynamical role of a variable n. Indeed, in a programming language, a variable stands for a cell in a computer's memory, so its dynamical meaning is self-evident. After some experience with operational representations, the student should no longer be confused by the double role a variable plays in algebraic representations.

2. The principle of mathematical induction is presented in "operational" terms (see Box 1(d)). While stressing the computational aspects of the concept of function, we can speak about equivalence of algorithms instead of dealing with equality of infinite sets. [Although the present version may seem somewhat restricted in comparison to the former one, it is in fact equally general. Indeed, any statement of the form "P(n) for every n∈N" can be transformed into a proposition on functions: "The characteristic function of P is equal to f, while f(n)=1 (TRUE) for every n∈N".]

The operational presentation is free from all the previously mentioned didactical disadvantages of the structural version. Firstly, the confusing proposition ∀n [P(n) => P(n+1)] practically disappears here under the cover of less formal (but by no means less exact) statement "f and g can be computed by the same recursion rule". This statement can be easily translated into appropriate actions. For instance, if f is the function presented in Box 1, and if $g(n)=n^2$, then the student has only to show that g(n+1) can be obtained from y=g(n) by the same recursion rule

$y+2n+1$, which - when applied to $y=f(n)$ - would yield $f(n+1)$. Since the confusing equalities $f(n)=g(n)$ and $f(n+1)=g(n+1)$ are not mentioned at all (the algebraic transformation $y+2n+1$ has to be applied to only one function at a time), there is no room here for the incorrect quantifications. Secondly, this time there is almost no danger of apparent vicious circle. The principle of induction has been phrased here as a meta-mathematical rule rather than as a mathematical axiom. Indeed, instead of dealing with a method of proving an equality of two infinite sets, we speak about a way of showing that $f(n_0)=g(n_0)$ for any given n_0. Thus, in our version the general quantificator has been transferred to the meta-language, and after this the inductive proofs would involve in fact only limited quantifications $(\forall n<n_0 \ldots)$.

3.Proving by induction. In our teaching unit the principle of induction is used only for proving equalities of functions (equivalence of algorithms). Other properties of numerical sequences (like those mentioned in problems 1 and 3 in Box 3, Test 3) are not explicitly dealt with. Hence, our coverage of the subject is not as broad as required by the curriculum. The entire unit, however, is meant for not more than 6-8 teaching hours (provided the students have some previous experience with programming language), so it can be incorporated into a regular course on induction as an introductory chapter.

COMPARATIVE EXPERIMENTAL STUDY

Our experimental investigation of the structural and operational methods of teaching is still under way. Some tentative conclusions, however, can be already drawn from the results of the pilot study, which will now be presented in some detail.

The experimental material on induction has been taught to four groups (56 students) in the Centre for Pre-academic Studies at the Hebrew University. After this six-hour introductory course, the pupils had to complete their training in the regular mathematics classes, where the subject was treated in the usual structural manner. The experimental groups have been compared to suitable control groups, in which induction had been taught only by traditional methods. In this comparison, three different tests have been applied (Box 2). Because of technical reasons, each test could be administered only to a part of the control groups.

Test 1: Recursion. The problem which was presented in this test was quite unusual in comparison to all the questions on recursion our students

BOX 2: The results of the experiment

TEST 1: Recursion

PROBLEM: Given $f(0)=0$, $f(n+1)=f(n)+n$, and $g(n)=f(n)+f(n+1)$ - find a recursion rule for $g(n)$.

RESULTS:

	Experimental group	Control group
N	19	19
no. of correct answers	13 (68.4%)	3 (7.4%)

TEST 2: The principle of induction

QUESTION: a. What are the main stages of an inductive proof for a claim: "$f(n) = g(n)$ for all natural n" (f and g are functions from N to N)?

b. How can you be certain that $f(n) = g(n)$ for any given n, if the equality of the two sequences has been shown by induction?

RESULTS:

	Exp. group	Contr. group
N	18	16
a: -correct answer	18 (100%)	4 (25%)
b: -"it is an axiom "	5 (28%)	12 (75%)
-"f(n) and g(n) are obtained from f(0)=g(0) by the same operations."	12 (67%)	3 (19%)

TEST 3: Proving by induction

PROBLEMS: 1. Prove that if $f(1)=2$ and $f(n+1)=f(n)+2^n$ then the last digit of $f(400)$ is 6.

2. If $f(1)=1$ and $f(n+1)=f(n)+(n+2)/3$, and if $g(n)=(n+1)(n+2)/6$, what is the truth set of the equation $f(n) = g(n)$? Prove this.

3. Prove that $\mathrm{mod}(4^n,3)=1$ for all n.

RESULTS:

	PROBLEM 1		PROBLEM 2		PROBLEM 3	
	E	C	E	C	E	C
no. of s's who participated	14	14	13	13	29	29
no. of s's who chose the problem	8	6	8	4	22	23
Average score (out of 10)	7	5.1	8.5	1.5	8.5	5.9
no. of s's who got maximal score	4	---	6	---	16	7

had met before. The figures in the table show, that in spite of the non-standard sequence definition, the experimental group was quite successful in finding the appropriate recursion rule ($g(n+1)=g(n)+2n+1$). In contrast, the majority of the control group failed in the task. It was quite clear that for the traditionally-instructed students, finding the recursion rule usually meant nothing more than writing any formula for $g(n+1)$. Indeed, many students wrote: "The recursion rule of g is $g(n+1)=2f(n)+3n+1$". Those who discovered (by help of numeric examples) the explicit formula $g(n)=n^2$ claimed that the rule is $g(n+1)=(n+1)^2$.

Test 2: The principle of induction. The results obtained on this test in the control group have been reported above. According to our expectations, the answers given by the experimental group were much more satisfactory. Literally all the participants of the experiment could restate the principle of mathematical induction (in its operational version), and most of them were able to explain it in quite convincing way.

Test 3: Proving by induction. The data presented here have been collected on three different exams in mathematics. Every one of our students participated in only one of these exams. The pupils had to solve four problems out of six. Only one of these problems dealt with induction. The question on induction which appeared in the first exam (Problem 1) was quite unusual for both experimental and control groups. The one which was given in the second questionnaire (Problem 2) was rather standard, although it was put in somewhat unconventional terms ("What is the truth set.."). The problems like the last one (Problem 3) did not appear in our experimental teaching unit on induction, but they were known to all the students from the regular course on induction. As can be seen from the data summarized in Box 2, the experimental group achieved significantly better results in both standard and non-routine problems.

CONCLUSIONS

It should be pointed out that the presented study suffered from certain technical shortcomings. Firstly, all our comparisons were based on rather small figures. Secondly, the experimental groups participated in both experimental and regular courses on induction, so they spent on the subject slightly more hours than the control groups. Even so, we have quite good reasons to believe that the unconventional method of teaching did contribute to the students' understanding of the subject. Indeed, since all the results indicated the same strong tendency, the general advantage of the experimental groups seemed to be undeniable; and since our tests contained mainly non-standard tasks, which required much more than technical skills, it is rather unlikely that this advantage was merely the result of the few additional hours of training. It remains to be seen if our future, better controlled studies will confirm these conclusions.

REFERENCES

[1] Avital,S. & Liebeskind,S., Mathematical Induction in the Classroom, Educ. Stud. in Math., v9, p429-38, 1978.
[2] Dubinsky,E., On Teaching Mathematical Induction,I, Journal of Math. Behavior, v5, n3, p305-17, Dec. 1986.
[3] Dubinsky,E., On Teaching Mathematical Induction, II, to appear.
[4] Sfard,A., Two conceptions of mathematical notions: operational and structural, Proceedings of PME-XI, Montreal 1987, v3, p162-69.
[5] Vinner,S., Implicit Axioms, Ω-rule and the Axiom of Induction in High-school Mathematics, The Am. Math. Monthly, v83, p561-66, 1976.

EPISTEMOLOGICAL REMARKS ON FUNCTIONS

Anna Sierpińska
Institute of Mathematics, Pol.Ac.Sci.

Abstract. The paper contains a tentative epistemological analysis of the notion of function both from the phylo- and ontogenetic points of view. The analysis is a part of a research aiming at elaborating didactical situations helping students to overcome epistemological obstacles related to functions and limits.

The paper presents a further part of research briefly reported in the XIth PME Proceedings (Sierpińska, 1987a). The research aims at elaborating didactical situations favouring the overcoming of epistemological obstacles related to functions and limits in 15-17 y.o. students (cf. Sierpińska, 1987b, 1985a). One of questions that such a research raises is the question of meaning of the mathematical concepts involved. This is the question we ask in this paper: we reflect on upon the epistemology of the notion of function.

I.- Epistemological obstacles related to functions

Re e.o. see Bachelard (1938), Brousseau (1977), Sierpińska (1985b).

We have distinguished several stages in the historical development of the notion of function (see Fig.1).

It seems that the development of the notion of curve contributed in many ways to that of the notion of function: it provided a context in which analytical tools for describing relationships could be developped. The beginnings of calculus were, in fact, linked with exploration of curves. Curves

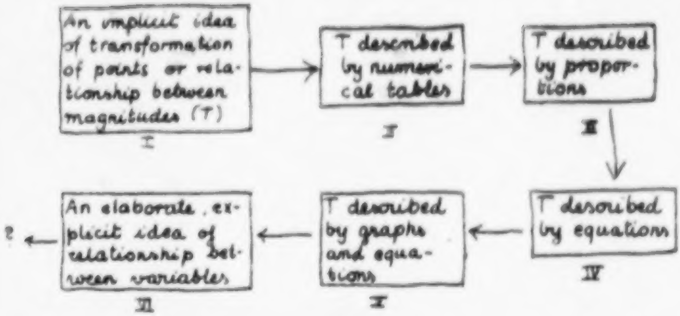

An implicit idea of transformation of points or relationship between magnitudes (T)
I
T described by numerical tables
II
T described by proportions
III
T described by equations
IV
T described by graphs and equations
V
An elaborate, explicit idea of relationship between variables
VI
?

Fig. 1 The historical stages of the notion of function (cf. Boyer, 'Proportion,... ')

were described by proportions between some auxiliary segments (diameter, axis,..) as in Fermat (Fig.2), or by equations between these, as in Newton (Fig.3). The system of auxiliary segments was chosen for every particular curve or class of curves separately: coordinates were not numbers determined by a system of coordinates chosen beforehand. They were segments - geometrical objects. Curves were not regarded as graphs of relationships between these auxiliary segments. They were taken for what they appeared to our eyes: geometrical objects or trajectories of moving points ("geometrical" or "mechanical").

We shall name this approach to curves "concrete" - insofar as it is based on direct data and contextual relations. Perhaps this "concrete" approach at curves was one of the most serious obstacles in the development of Calculus. Some forms of this obstacle seem to be still present in today's students.

II.- Students' conceptions of functions

Three groups of 15-17 y.o. students were involved in the research. Here we shall refer mainly to conceptions of 4 humanities students: Agnès and Ewa (17) and Darek and Gutek (16). The students underwent a series of sessions composed of different didactical situations. A didactical situation is characterised, among others, by a social context, type of teacher interventions and a mathematical context. In our research, the mathematical context was based on the topic of properties of fixed points of functions (Engel, 1979). Social contexts such as working in small groups, communication of meaning between students were used. Negotiation of meaning, suggestion of a way of search, discussion with students are examples of our interventions.

In Poland, the notion of function is introduced to 13 y.o. in its very elaborate abstract form. But the general definition is so comprehensive that it says nothing to children that know very little mathematics and even less physics. Children are given examples and different symbolic and iconic representations are shown to them. It is on this material that they build their meaning of the term "function" and more often than not this meaning has nothing or very little to do with the most primitive but fundamental conception I of function (a relationship between variable magnitudes). A student's conception of function can be a complex (in the sense of L.S.Wygotski) composed of one or more degenerated forms of the historical conceptions II- V. These forms may well function parallelly without there being any conscious link among them.

We have divided the students' conceptions of functions into two main categories: "concrete" and "abstract" (Bernstein, 1971). In these, further distinctions are made (see Table 1).

Concrete conceptions of functions in students

- mechanical (CM-f) := a function is a displacement of points (in non-verbalised versions this conception corresponds to the historical stage I);

-synthetic geometrical (CsG-f) := a function is a "concrete" curve, i.e. a geometrical object, idealization of a line on paper or a trajectory of a moving point;

-algebraic (CA-f) := a function is a formula with "x" and y" and equality sign; it is a string of symbols, letters and numbers;

Abstract conceptions of functions in students:

-numerical (AN-f):= a function is a transformation of some things into other things; these new things or their position can be described by numbers (the values of the function); a function is given by a sequence of its values. This conception ressembles the historical conception II but it may be vague or implicit in the student's mind; in particular, the necessity of naming the parallel sequence of arguments may not be felt at all;

-algebraic (AA-f) := a function is an equation or an algebraic expression containing variables; by putting numbers in place of variables one gets other numbers; the idea that the equation describes a relationship between variables is absent here. The conception is a degenerated form of the historical conception IV (stage IV without stage I);

-analytic geometrical (AaG-f) := a function is an "abstract" curve in a system of coordinates, i.e. the curve is a representation of some relation; this relation may be given by an equation and curves are classified according to the type of this relation (first degree, algebraic, transcendental,..); it is not the relation that is called function: it is the curve itself. This conception is a degenerated form of the h.c. V;

-physical (APh-f) := a function is a kind of relationship between variable magnitudes; some variables are distinguished as independent, other are assumed to be dependent of these; such relationships may sometimes be represented by graphs. This is close to the h.c. VI.

The APh-f was not observed in any of the students. We have added it here, however, because we think that such a conception is attainable by students of this level (indeed it is implicit in their conceptions but it is not this that they would call "function") provided that appropriate mathematical contexts are used to develop it. Agnès was quite close to it. The context of attractive fixed points of functions, especially if extensive use of graphical representations is made, proved to favour the AaG-f and seemed even to create obstacles to the development of the desired conception VI.

To better be able to analyse the students' speech events we have constructed a "frame" for the definition of an attractive point of a function. The frame divides a possible definition into parts each of which answers a particular question. The first question is: "What is the domain of our investigation?". Students' response to this question allowed us to make inferences regarding their conceptions of function.

A posteriori we have established a table of students'answers to these questions. We have scored these answers, the maximum score being attributed to the best of all students'answers in all cases except the first question mentioned above. These were scored as shown in Table 1. The scores were not used to evaluate the answers as right or wrong. We just needed a tool allowing us to detect in a more objective way moments of important conceptual "jumps" and thus judge of the influence of particular social contexts and interventions. The scores in vulgar fractions can also be used as codes for answers.

Table 2 shows the students' conceptions of functions as they developped through different social contexts. Further research consisted in close analysis of moments where high conceptual jumps seemed to be made. For example, Agnès made her great jump in the social context of work in small groups under the influence of criticisms of her group-mates. At the start, Agnès'conception of function was CM-f. While explaining her ideas of solving the problem she gradually developped tools for analytical representation of relationships between the varying distances of moving points from the fixed point. But she refused equations of the form "y equals". She preferred proportions. Agnès seemed to be recapitulating the history of the notion of function. Later on, while working on a written communication of the concept of attractive fixed point to a class-mate, she displayed an interest in numerical approximations of terms of sequences $x_0, x_{n+1} = f(x_n)$ that were included in materials she and her group received. She made right inferrences about the ratio in which the sequences were increasing or decreasing.

III.- Final remarks

1. The most fundamental conception of function is that of a relationship between variable magnitudes. If this is not developped, representations such as equations and graphs loose their meaning and become isolated one from the other. A deviation from the genetic line is made. Introducing functions to young students by their elaborate modern definition is a didactical error - an antididactical inversion (cf.Freudenthal, 1983).

2. The context of attractive fixed points of functions introduced with heavy use of graphical representations does not help to develop the above mention ed fundamental conception of function. It is too geometrical-algebraic. A context of physical magnitudes and various relationships between them would probably be better. This demands a cooperation between the mathematics and the physics teachers.

References

Bachelard, G.: 1938, La formation de l'esprit scientifique, PUF, Paris

Bernstein, B.: 1971, Class, Codes and Control, London

Boyer, C.B.: ? "Proportion, equation, function: three steps in the development of a concept", ? (I am in possession of an nth, n →∞ , xerox copy of the paper)

Brousseau, G.: 1977, "L'étude des processus d'apprentissage en situations scolaires", Communication à l'Université de Leeds, 77-07-06

Engel, A.: 1979, Mathématique élémentaire d'un point de vue algorithmique, CEDIC, Paris

Freudenthal, H.: 1983, "Implicit philosophy of mathematics history and education", Proceedings of ICM-82, Warsaw, PWN

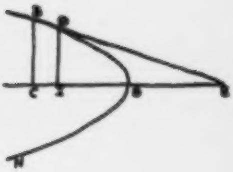
Marody, M.: 1987, Technologie intelektu, PWN, Warszawa

Sierpińska, A.: 1985a, "Obstacles épistémologiques relatifs à la notion de limite", Recherches en Didactique des Mathématiques, 6.1, 5-67

Sierpińska, A.: 1985b, "La notion d'obstacle épistémologique dans l'enseignement des mathématiques", Actes de la 37e Rencontre CIEAEM, Leiden 1985

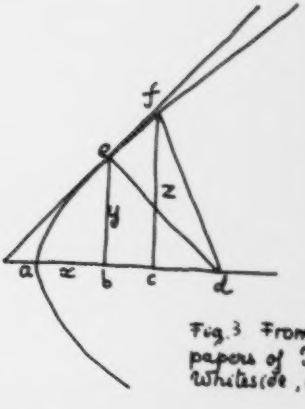
Sierpińska, A.: 1987a, "Attractive fixed points and humanities students", Proceedings of the XIth PME, Montreal, v.III, 170-176

Sierpińska, A.: 1987b, "Humanities students and epistemological obstacles related to limits", Educational studies in Mathematics, vol, 18, 371-397



Soit donneé, par example, la parabole BDN, de sommet D, de diamètre DC; soit donné sur elle le point B par lequel il faut mener la droite BE tangente à la parabole et rencontrant le diamètre en E. Si l'on prend sur la droite BE un point quelconque O, dont on mène l'ordonneé OI, en même temps que l'ordonneé BC du point B, on aura: $\frac{CD}{DI} > \frac{BC^2}{OI^2}$ puisque le point O est extérieur à la parabole. ...

Fig. 2 From "Oeuvres de Fermat", Vol. II, Paris, Gauthier-Villars et Fils MDCCCXCVI, p. 122



Let ab = x, eb = y,
bd = v, bc = o, cf = z,
ed = df.

Let y^e nature of the crooked line be:

$$ax + xx = yy$$

Fig. 3 From "The mathematical papers of Isaac Newton", Vol. I Whiteside, ed.

Table 1. Students' conceptions of functions and their scores

Legend: α β γ δ α: conception which is non-verbalized; β: ... verbalized in a restricted code; γ: ... pseudo-elaborate code; δ: ... elaborate code

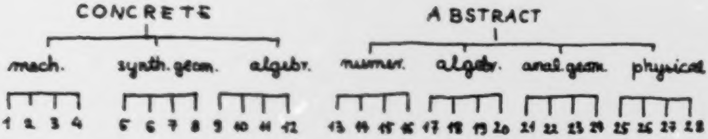
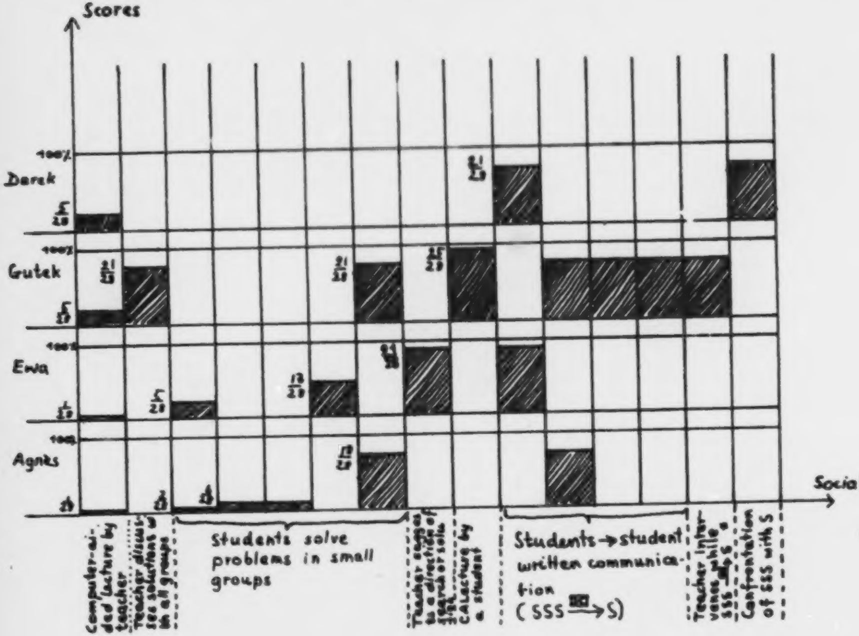

CONCRETE
ABSTRACT
mech.
synth. geom.
algebr.
numer.
algebr.
anal. geom.
physical
1 2 3 4
5 6 7 8
9 10 11 12
13 14 15 16
17 18 19 20
21 22 23 24
25 26 27 28

Table 2. Development of students' conceptions of functions



(*) The intervention consists in
(i) suggesting a direction of search or solution
(ii) negotiating a meaning

FORMATIVE EVALUATION OF A CONSTRUCTIVIST MATHEMATICS TEACHER INSERVICE PROGRAM

Martin A. Simon, Mount Holyoke College

The Educational Leaders in Mathematics Project was designed to assist secondary mathematics and elementary teachers in developing a constructivist epistemology as the basis for mathematics instruction. The Project provides teachers with an intensive two-week summer institute and a full academic year of weekly classroom supervision. Formative evaluation, two and a half years into the Project, suggests that (1) these two components result in significant classroom changes, (2) teachers' classroom implementation efforts can be described by one of four patterns, and (3) some important training and support needs of the teachers are not met by this structure.

INTRODUCTION

SummerMath for Teachers' Educational Leaders in Mathematics Project (ELM) at Mount Holyoke College is an inservice program for elementary teachers and secondary math teachers. The program is designed to (1) assist inservice teachers in developing a constructivist approach to mathematics instruction (Mundy, Waxman, and Confrey 1984), and (2) to develop teachers as workshop leaders to introduce their colleagues to a constructivist approach to mathematics instruction. This report will focus on the first of these two goals.

PROJECT DESIGN

Following is a description of the ELM Project. For the purpose of this report, we will focus on the first three stages (out of five), the stages which are most directly related to the inservice development of the participating teachers.

Stage One: Summer Institute Two two-week institutes (one for elementary and one for secondary) provide an introduction to constructivist mathematics instruction. Participating teachers experience

---

** Work supported by National Science Foundation Grant #TEI-8552391

the role of student in a constructivist classroom, constructing mathematical concepts which are new and challenging for them (maybe familiar concepts, but explored in greater depth). They also focus on children's learning of mathematics and work on their ability to ask probing questions and to design sequences of constructivist lessons.

Stage Two: Academic Year Follow-up Teachers participate in the follow-up program from September through May following their involvement in the summer institute. An ELM staff member meets on a weekly basis with each participating teacher in that teacher's classroom. During the math class, the staff member either observes the teaching of the participating teacher or provides demonstration teaching. Following the math class, the teacher and ELM staff member meet to discuss what happened during the math lesson, to informally evaluate the learning, and to discuss possible next steps. Each teacher chooses those aspects of the summer's work that she wants to work on implementing. During this academic year, teachers also meet with their ELM colleagues and Project staff in four workshops in which further work is done on developing constructivist instruction, and discussions take place between teachers about implementation successes and difficulties.

Stage Three: Advanced Institute The Advanced Institute is designed for teachers to deepen their knowledge and understanding of constructivist math instruction and to further develop their teaching skills. The institute begins, once again, with an opportunity for the teachers to experience the role of learners of mathematics. A far greater portion of this institute is spent in the development and critiquing of constructivist lessons.

STRENGTHS OF THE PROGRAM

Feedback from participating teachers has helped us to identify several strengths of the program:

1. In the summer institute, teachers construct their own concept of constructivist education. Through reflecting on their own learning of mathematics and the learning of children, teachers reorganize their internal models of mathematics instruction. Teachers have written:

> As the week has progressed, my conceptions of how mathematics is learned have changed daily, sometimes even

> hourly. I know that what I think and feel now is not the total picture or a final answer.
>
> As I participate in this institute and experience first-hand the growth of my own mathematics..., my conceptions have had to start to change to resolve the conflict of my previous beliefs and the techniques I have seen work this week.
>
> After this week, I discovered that my most meaningful learning experience was not when I was on the correct path, but when I was off on a tangent that led absolutely nowhere. I out of ignorance have almost consistently prevented this type of valuable learning experience from happening in my classroom.

The opportunity for teachers to construct their own understandings about mathematics learning and teaching results in teachers' personal commitment to implement their learnings and teachers' sense of control over the changes to be made.

2. The follow-up program: Teachers have reported that they value the moral support, the opportunity to discuss difficulties as well as successes with ELM staff and colleagues, the modeling of demonstration lessons in their classrooms, and the help in critiquing lessons and thinking about next steps. The consistency of the structure, knowing that a staff member would be there every week, prevented their putting implementation efforts on the back burner. Teachers commented:

> It is every week. I enjoyed the chance to reflect on what has been going on. It provides me with a focus, a time to set aside for thinking about what I want to accomplish, and how to determine if that happened. Without the weekly meetings, I fear the time would be spent doing other things.
>
> My consultant keeps me fresh, provides alternatives when I have run out, puts the issues in a different perspective, provides an excellent model for questioning skills.
>
> I like best the support of the consultant and the ongoing motivation that she provides. Without the follow-up program, I would not have had the stamina to continue."

The major commitment of consultant time and financial resources required to carry out a follow-up program of this scope seems to be necessary for successful implementation of constructivist principles.

3. Teachers valued the chance to return for additional summer work following a year of classroom implementation. The most consistent comment

that we heard is that during the Advanced Institute, previous learnings "really seemed to come together." Teachers wrote:

> The Advanced Institute is most important because you have one year's experience to draw upon when you arrive, and many questions and concerns. I feel that I have internalized many of the behaviors that I had been approaching rather tentatively.
>
> Learning to teach math using a constructivist and problem solving view is an overwhelmingly difficult and expansive undertaking. In no way is a two-week institute adequate in helping us develop our understanding of how students learn and guiding us in making the necessary changes in how we teach. Actually, I think I would profit from coming to the Advanced Institute any and every summer.

ASSESSMENT AND CHARACTERIZATION OF IMPLEMENTATION

Assessment of Implementation: The Levels of Use (LoU) structured interview (Hall, et al 1975) was used with each of the teachers at the end of the follow-up program to determine the extent to which they had implemented a constructivist approach to instruction. The LoU interviews are scored by assigning one of the following levels:

Level 0 - nonuse
Level I - orientation
Level II - preparation
Level III - mechanical use
Level IVa - routine
Level IVb - refinement
Level V - integration
Level VI - renewal

As the Project proceeded, we settled on a refinement of the LoU scoring to better differentiate among the various implementation efforts of our teachers; separate Lou levels were determined for teachers' implementation of "constructivist teaching strategies" and for the level of implementation of a "constructivist epistemology."

Teachers who implemented "strategies" chose to use one or more tools of constructivist teaching because of their perceptions that these tools would contribute to their students' learning. (Eg. "I have been asking probing questions, because it is important that my students think about why things that they do work.") Teachers who implemented strategies may not have had a sense of the part that these strategies can play in facilitating the construction of mathematical understanding. The strategies that teachers identified and which emerged as significant in the program were the following:

- -Use of non-routine problems
- -Use of Logo, the Geometric Supposers or other computer tools for exploration
- -Use of manipulatives and diagrams
- -Exploring alternative solutions
- -Problem solving in pairs and groups
- -Use of probing (non-leading) questions
- -Providing wait time
- -Asking for student paraphrasing of other's ideas
- -Pursuing thought processes following right and wrong answers

The teachers who were judged to have implemented a "constructivist epistemology" saw the strategies as serving the larger goal of construction of mathematical understandings and consequently made decisions on if and when to use particular strategies based on whether this larger goal would be served. These teachers tend to be more concept-oriented and more self-sufficient in generating ideas for instruction and evaluating the results of instruction.

Lou scores for the 1986-87 ELM teachers indicated that all twenty-eight had implemented at least one strategy at Level III or higher and twenty-five at Level IVa or higher. Twenty-one had adopted a constructivist epistemology at Level III or higher, nineteen at Level IVa or higher.

Characterizing Implementation: Combining the LoU ratings, which described only the implementation level in May, with weekly observations throughout the follow-up year, four patterns of implementation became clear.

1. For some teachers the combination of their previous experiences/ideas and their experiences in the summer institute resulted in the adoption of a constructivist epistemology from the beginning. Such teachers described themselves in the following ways, "I knew that based on what I saw and understood this summer that I had to completely change mv approach to teaching." and "The night before the first day of school I was paralyzed, I couldn't just teach the way I had in the past. I knew what I wanted to do, but I didn't know where to begin."

Teachers in this group began, sometimes awkwardly, to develop lessons that focused on student construction of concepts. Throughout the year, working with an ELM staff person, they refined their efforts.

2. Some teachers chose to integrate particular teaching strategies (eg. wait time, probing questions, group work, use of manipulatives) into

their traditional ways of teaching. Some of these teachers never progressed further. They were pleased, sometimes excited about the benefits that they perceived from using these strategies and continued to include these strategies as a regular part of their teaching. Others of these teachers, through their work with ELM staff and regular analysis of student learning, were able move from the implementation of isolated strategies to the development and implementation of a constructivist epistemology.

3. Some teachers characterized their efforts as "doing SummerMath" once or twice a week. At these times, they used non-routine problems and/or manipulatives, they asked probing questions, refused to give the answers to the problems, and often had the students working in groups. They seemed to believe that these types of experiences were valuable for enrichment. However, they considered it separate from the curriculum that they were supposed to "cover."

As the follow-up year progressed, some of these teachers began to see connections between "doing SummerMath" and the curriculum. Seeing the understandings that were developing as a result of the new strategies, they began to see how aspects of this work could enhance or replace the curriculum work that they were doing. For some the result was the development of a constructivist epistemology.

4. A few teachers seemed to employ one or more strategies once a week when the ELM staff person was there because they felt that was expected of them. The lack of personal commitment was generally an obstacle. However, occasionally positive response on the part of the students persuaded the teacher of the value of one or more of the strategies.

LIMITATIONS OF THE PROGRAM

In observing teachers in the classroom, talking to teachers and reading their written feedback, a number of limitations of the current program have become clear.

(1) Many of the elementary teachers are limited by their own understanding of mathematical concepts, and of mathematical thinking in general. Many of these teachers were not successful as mathematics students and took very little mathematics. Those who are developing teaching strategies to help students discover important mathematical

concepts are feeling the limitation of not understanding the concepts further, or not having more insight into the interconnections of different mathematical concepts. Many of them express the feeling that during the summer institutes, they had their first taste of success in mathematics and a feeling that they could learn to understand mathematics. They express a willingness to study more mathematics, but they are looking for an opportunity to study mathematics taught using a constructivist approach.

(2) Because of the greater complexity of mathematical concepts taught in the secondary schools, secondary teachers struggle more than elementary teachers in designing concrete activities as a foundation. Also, they are often unable to do the task analysis necessary to identify subconcepts and connections with prior concepts.

(3) Both elementary and secondary teachers, while novices in constructivist teaching, are being put in a situation of having to create their own curricula. This is an overwhelming task, only somewhat mediated by the support of the ELM staff member during follow-up. Constructivist teaching requires a certain amount of creation on the part of the teachers, but does not require teachers to invent everything from scratch. Curriculum materials consistent with constructivist teaching must be developed, and materials and references that are concept-based rather than topic-based, as in conventional textbooks, must become available.

(4) The higher the grade level, the more frustration and conflict the teachers experience because of the weak conceptual foundations of their students. The constructivist teacher who spends more time listening to students, evaluating their understanding, and creating activities which allow them to build on previously firm understandings, come into contact frequently with the huge gaps in understanding that students have. Whereas the primary school teacher may be comfortable working on concepts that should have been learned a year or two before, the high school teacher, faced with students who need a course in fractions or ratio but find themselves in an algebra II or trigonometry class, experience a great amount of conflict between the schools' expectations of what they should teach and their awareness of what their students actually need.

CONCLUSION

The ELM Project has demonstrated the power of combining a constructivist summer institute experience with an intensive follow-up program. Classroom implementation ranged from the incorporation of new and powerful teaching strategies to the construction of mathematics programs based on constructivism. The Project's work has also highlighted some of the unmet needs of teachers which prevent them from functioning more fully using a constructivist approach. The identification of these needs can inform and direct future efforts.

The extent to which ELM teachers were able to develop and implement a constructivist approach varied greatly. This large variation can be attributed to characteristics of the teachers prior to entering the program. The relationship of teachers' characteristics (pedagogical schema, attitudes, beliefs, and personal factors) to the development of a constructivist approach to instruction is poorly understood and needs to be investigated.

REFERENCES

Hall, G.E., Loucks, S.F., Rutherford, W.L., and Newlove, B.W. Levels of use of the innovation: A framework for analyzing innovation adoption. The Research and Development Center for Teacher Education, Spring, 1975.

Mundy, J., Waxman, B., and Confrey, J. A constructivist/cognitive process perspective for mathematics teacher education. Paper presented at the Annual Meeting of AERA. April 1984.

CONSTRUCTION AND RECONSTRUCTION: THE REFLECTIVE PRACTICE IN MATHEMATICS EDUCATION

Beth Southwell
Nepean College of Advanced Education

An initial study was carried out to investigate the relationship between experience and reflection on that experience. In this study third year teacher education students were asked to prepare a structure of meaning diagram using problem solving as the focus. Students' responses were encouraging, and indicated that they perceived the technique as being valuable in helping them to synthesise isolated understandings and prompt connections not previously seen.

The use of the structure of meaning technique has been further refined in the light of the pilot study and applied to a more closely defined area of problem solving. It has also been applied to other areas, namely geometry and measurement.

This investigation into the relationship between actual experience and reflection on that experience was extended to another technique. The one chosen was the repertory grid by which subjects were encouraged to explore their own thoughts and feelings in relation to their problem solving program. The technique relies on subjects establishing poles at either end of a continuum and comparing elements of the subject with these poles.

The students found that constructing the grid following a fairly structured procedure was a valuable task in itself. According to their reports the completed task was even more valuable.

Some attempt is made to evaluate this and other reflective practices in the process of learning.

One of the critical issues in learning mathematics which none of the psychologists seem to have adequately covered is the balance between theory and practice or the interplay between experience and actual acquisition of concepts. The reflection upon the problem solving process is a key element in learning through problem solving. Techniques devised to enhance the reflective process need to be applied to mathematical problem solving and to mathematics education research.

Several have emphasised the role which reflection has in learning mathematics. Skemp (1979, 1980) makes a distinction between intuitive and reflective intelligence. Kilpatrick (1984) in his address to ICME V in Adelaide stressed its importance. Wittmann (1985) has developed a theory of reflective practice. Burton (1984) and Mason (1986) apply reflection to the problem solving process.

REFLECTIVE PROCESSES

Despite the general acceptance of the necessity for reflection for learning to be effective, not many have attempted to define or describe reflection or to develop reflective strategies.

The processes involved in re-evaluating experience are association, integration, validation, and appropriation. New ideas need to be associated or connected with what we know already. Then associations need to be integrated into a new whole in an organised way. What we have started to integrate must be validated or tested for such things as internal consistency and for consistency between our new ideas and those of others. Then for some, though not all, learning tasks, we need to allow them to enter into our sense of identity and become part of our value system. Commitment to action is then possible and should follow.

Strategies to help learners to reflect on their experience are varied. Some have been in use for a long time, though not always in mathematics. Such simple procedures as discussion, keeping logs or conversation, while recognised as valuable, are not often used consciously to assist the reflective process. Several more dramatic strategies have been developed at the Centre for the Study of Human Learning at Brunel University. One of these is the Structure of Meaning Technique. Its purpose is to help a learner reflect on how he or she is structuring new knowledge. It allows learners to depict diagrammatically what they consider important items of meaning.

A REFLECTIVE STRATEGY IN ACTION

It seemed to the writer that this would be a useful technique to employ with trainee teachers both as a means of clarifying and integrating their own knowledge, but also as a model for modified use in school. Third year students at Nepean College of Advanced Education were completing a sequence of three courses in Mathematics, during which problem solving and mathematical investigations had been stressed. Thirty three students agreed to participate in a brief study to test out the effect of applying the Structure of Meaning technique to the area of problem solving. They were not given any warning as to when the process would be applied, hence had no opportunity to do any preparation.

The students were in two groups, the first of thirteen, and the second of twenty. The task was explained to them in terms of constructing a diagram linking critical aspects of their understanding of the process of teaching problem solving. The first group were given a simplified version of a Structure of Meaning diagram with an example of how it might relate to problem solving. The second group was given the simplified version but not a specific example. They were, however, given the suggestion that they might find it worthwhile to list some of the critical aspects first before trying to put them into the diagrammatic form. They were all asked to construct the diagram, then explain it to their neighbour. The final part of the task was to write down how they felt about the task, what was good about it, what was not good and how it helped them - if it did.

The students responded well to the task. Some found it difficult to get started and their final products were not as sophisticated as they might have been, but everyone expressed their feeling that it was a beneficial process. The following are some of the reasons given:

> "It was a form of revision."
>
> "It made us think for ourselves."
>
> "The procedure was helpful in culminating thoughts on problem solving."
>
> "It reveals the importance of teaching being organised in a logical sequence."
>
> "It shows how skills learnt in other areas of the curriculum can be used and are necessary for problem solving."
>
> "It drove home the inter-relatedness of aspects of problem solving - making it clearer to view problem solving as a process in totality, rather than a number of discrete aspects."
>
> "This was helpful in providing the opportunity for me to evaluate my own ideas about what the aspects of teaching are and the inter-relationships between these aspects. Through reflection on my previously held knowledge about the aspects of problem solving I am now more sure about the needs of the children when learning problem solving."

The example given by the writer appears to have influenced the line of approach taken by a number of students in the first group in that their diagrams included the three approaches to teaching problem solving, namely, teaching for problem solving, teaching about problem solving and teaching through problem solving.

The second group completed a list of ideas before putting them into their diagram. The lists were quite extensive, but the diagrams were more limited than the first group. As the point of the exercise was to give the students experience in a technique which they might find helpful in their study, the "quality" of the diagram is of little importance.

Several suggestions were made by the participants to improve the effectiveness of the exercise. Some felt they were handicapped by not having their lecture notes with them, while those who did, felt the strategy helped them revise their notes. One student felt a whole class discussion would have been helpful. Several expressed the need for a starting point, though at least one said it was better not to be given much direction. The second group reported that they had felt unsure of the task at first but when they got going, they found it very helpful.

While the study appears to indicate that the Structure of Meaning technique can be applied to programs in mathematics education in pre-service courses, and does achieve positive results in that the participants admit to being encouraged to think because of it, there were still one or two who wanted others to do their thinking for them. This can be seen in this report:

> "I think it would be a good idea to have you write what you think the main aspects of problem solving are on the board in case someone has the wrong idea."

A further group of subjects were asked, not only if they were willing to participate, but also when they would be ready to carry out the task involved. The task itself was only explained briefly in the recruitment stage, but sufficient information was given to alert the subjects to the possible need for bringing notes and any material they wished to refer to. As it happened, they felt that their notes were all they needed on the occasion, though some did express their wish to follow up certain elements that arose as a result of the activity.

(a) Probem Solving

The subjects who participated in the second Structure of Meaning Activity were from two different sources. The first group were a small group of five secondary teachers who attended a five hour in-service workshop on problem solving organised by the Mathematical Association of New South Wales. During the workshop, participants, twenty five in number, were given the opportunity to actually work on problems of their choice in groups, then reflect on the processes and strategies they had used in solving their problems. Input from three speakers included theoretical and practical ideas both for solving problems and for implementing a problem solving approach in the classroom. The five who volunteered to contribute a structure of meaning diagram were all secondary teachers though there were a few primary teachers present.

Again, comments by this small group endorsed those made by participants in the previous study.

(b) Geometry

Another group of subjects to use the structure of meaning technique was taken from the third year students at Nepean College of Advanced Education. These students were within a week of completing their sequence of three mathematics course units and during the last of these had been concentrating on problem solving, geometry and measurement. Thirteen subjects in this study worked on geometry and twenty three on measurement.

In the geometry section, subjects were given a brief explanation of the structure of meaning diagram, and an example. They were then asked to list the elements of geometry which they considered should be covered in Years K - 6 before putting them into their diagram. It was interesting to note that while most students listed concepts or topics, a few listed general principles, such as the importance of using environmental instances or examples.

As before, the diagrams varied considerably. There were three who used a central focus, e.g. shapes, while three others strung ideas together in a sequential manner. The remaining subjects drew diagrams which included some clusters, and some sequences. The diagrams are a powerful evaluation of the process and, as such, provide very useful data for course development.

Again, the comments of the participants in evaluating the technique are most interesting.

> "It is a good strategy for refreshing me with the knowledge that my grasp of geometry is abysmal."

> "This exercise is good for revision of the concepts and their relationships to each other. It helps to bring together concepts and in doing so how they can be studied."

Subsequently, a small number of the subjects have indicated that they did in fact use the technique when preparing for their examinations. One used it as a means of determining her weak areas, the others as a means of structuring content.

(c) Measurement

The task relating to measurement was structured rather differently from the geometry task. Subjects were presented with a series of written situations and asked to extract from them the basic principle they would need to remember when providing measurement activities for children. These basic principles dealt with conservation, developmental levels, the importance of "hands-on" experiences, estimation and, the use of informal measures. They were then asked to draw a structure of meaning diagram using these basic principles.

The diagrams drawn indicated that the subjects saw measurement as a series of basically unrelated activities, and consequently the activity proved to be a very good diagnostic instrument. The insights gained by the writer as a result of the subjects' diagrams enabled her to prepare workshop activities to present some structured ideas.

REPERTORY GRID TECHNIQUE

The Repertory Grid is a means of providing subjects with a way of recording their understandings about some part of their environment or thinking. The subject on this occasion was asked to name a range of elements in teaching problem solving. These elements were then written on cards and the subject was presented with three of them at a time and asked to decide which of the three were the most alike. The subject was then asked to say why they were alike and why the third unselected card was different. In this way, poles of the construct under consideration were established. Once the poles were established, the subject was asked to rank all the other elements along that construct continuum. The procedure was repeated using a different set of three elements until all of them had been used.

At the conclusion of the process, the subjects were asked to reflect on the process itself and write some comments about it. They responded as follows:

> "This technique was good as it made me think more deeply about what I would do when preparing activities for children. I think I tend to work the other way around, find

> an activity then think of a way to teach it. I don't think I consciously plan it the way we just did it, but if I think about it, it would probably be the best way."

> "This has certainly made me think about words and how you can write down what you think are two very different ideas, but when pointed out to you are similar. This can be of great help when wording problems or even writing reports or keeping records for various subjects. Has made me think about what I have been doing with problem solving."

These comments seem to indicate that the technique is a useful one in providing a means of recording thoughts and feelings. The benefit possibly comes from the necessity of sifting carefully through the similarities and differences involved. The drawback is the length of time it takes to present the procedure individually to each subject.

RETELLING

Thirty four subjects from the third year teacher education program participated in the Retelling activity in two groups of seventeen subjects each. A statement concerning the use of calculators in the primary school was introduced to the groups, but, before they actually received it, they were asked to predict, on the basis of the title alone, what it was likely to be about. This tended to raise issues in their minds and enabled them to state explicitly their existing knowledge or lack of it. Thus it became a means of diagnosis. The subjects were told the purpose of the retelling, then given the paper to read. Two purposes were suggested and the groups were told they could choose whichever appealed to them. The two purposes were, from the viewpoint of a teacher, to convince the executive of the school to buy a set of calculators for the class, and to convince an uneducated parent about the value of using a calculator in mathematics to develop concepts, etc. At this stage, they were working in pairs or a group of three. Having read the paper, they were then to retell orally to each other the content and spirit of the paper, taking on the role they had selected. Finally, they were asked to write their arguments in whatever role they had assumed, and to evaluate the process in terms of its potential for assessment and learning.

REFERENCES

Boud, David, Keogh, Rosemary, Walker, David (eds) (1985) Reflection: Turning Experience into Learning, London:Kogan Page

Burton, Leone.(1984) Thinking Things Through.Oxford:Blackwell

Mason, John (1986) "Challenge to Change:'To Stay Alive, Every Teacher Must be a Researcher". A.A.M.T. Proceedings. Brisbane, A.A.M.T.

Novak, J D and Gowin, D B (1984) Learning How to Learn London:Cambridge Univ Press

Resnick, Lauren and Ford, Wendy (1984) The Psychology of Mathematics for Instruction. London: Lawrence Erlbaum Associates

Skemp, R R (1979) Intelligence, Learning and Action. Chichester: Wiley

Skemp, R R (1980) The Function of Intelligence and the Understanding of Mathematics. Paper presented to ICME 4, Berkeley

Wittman, E. (1985) Practising Skills and Reflection. Paper presented to 37th C.I.E.A.E.M., Netherlands, 1985

The evaluations of the procedure indicated that the students found it very helpful in sorting out their ideas and coming to a position about the use of calculators in the primary school. A sample of the subjects' comments follows:

> "The practical activity showed that children can be motivated to do maths and that maths can be fun.
>
> Role playing the teacher, the executive teacher, and the innumerate mother, gave an interesting perspective on the use of calculators. It made us (me) think of the practical advantages of children using calculators and also necessitated a framing of my own attitude.
>
> There is much more to, and more advantages of, using calculators than I thought.
>
> Role playing was much more relevant than straight exposition on their use. I had to empathise with the teacher, child, mother and executive."
>
> "As a parent, I have myself questioned the use of calculators. However, the exercise we did yesterday made me think about the potential of calculators and having seen young children play with them, I know that they hold a great deal of fascination for children. The exercise put me in a situation I may well be in one day as a teacher and helped me to order my thoughts and develop an opinion."

ACTION RESEARCH IN REFLECTION

This study of reflection in action calls for further reflection. Further reflection is needed to improve the effectiveness of the implementation of the strategy used. It is also needed to assess or evaluate the effectiveness of the strategy and the reflection, and to plan for further stimuli to promote reflection. In this, reflection ceases to be a purely individual activity and becomes a social act.

The three techniques considered all have a value in mathematics education There is, as yet, insufficient evidence to claim that they are all equally valuable for all branches of mathematics. Many variations are possible, so it could be that they can all be adapted to suit the subject matter. This in itself would be an effective reflectional procedure. If ways of introducing these procedures, and others of a similar nature, could be found, students at all levels would benefit from their use.

Commitment to action is one of the outcomes of the reflective processes. If students of mathematics are encouraged to reflect on their experience, either in completing exercises or in solving problems, learning will result.

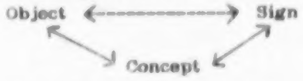
GRAPHICAL LESSON PATTERNS AND THE PROCESS OF KNOWLEDGE DEVELOPMENT IN THE MATHEMATICS CLASSROOM

HEINZ STEINBRING & RAINER BROMME
INSTITUTE FOR THE DIDACTICS OF MATHEMATICS (IDM)
UNIVERSITY OF BIELEFELD F.R. OF GERMANY

A method for analyzing mathematics teaching is presented which permits to take into account the different levels of mathematical meaning within teacher-students interactions. Conceptual structures of the development of mathematical knowledge are visualized by means of graphical diagrams.

1. The construction of meaning in mathematics teaching

The meaning of mathematical knowledge cannot be established in teaching processes by formal definitions of concepts alone; meaning is developed, negotiated, changed and agreed upon in interaction between teacher and students. On the one hand, the joint construction of meaning depends on socio-communicative conditions of teaching processes (cf. Bauersfeld 1982, Voigt 1984); on the other hand, the epistemological nature of mathematical knowledge fundamentally influences the construction of meaning. Meaning is not immediately "included" in the symbolic representations of knowledge; the meaning of a sign-system is contained in its "intentions", its use or its reference to an "objective" situation. Accordingly, the meaning of a mathematical concept is conceived of as a relational-form which has to be established between 'sign' and 'object' in the epistemological triangle:



(cf. Mellin-Olsen 1987, Ogden/Richards 1923, Otte 1984, Steinbring 1985, 1988). Meaning as a relational-form, in particular, means that knowledge realizes an objective relation between 'sign' and 'object', a relation which simultaneously has to be constructed by the learning subject and which has to be agreed upon with others in communication.

On the basis of this epistemological triangle, a schema of coding was constructed for analyzing the development of mathematical knowledge in the (verbal) interaction of teaching. The novel type of a graphical visualization for the coded data (cf. the diagrams A1, A2, B1, B2) in form of an "epistemological cardiogram (ecg)" shows global patterns and specific structures of knowledge development.

2. The coding of transcribed lessons

From a sample of 26 teachers, two are chosen (teachers A and B) who showed significant differences with regard to individual variables of the quality of teaching in lesson observation (for further details see Bromme/Steinbring 1987). For each of these teachers, two transcribed lessons (in the 6th form) introducing stochastics are analyzed. According to the epistemological triangle, the statements of students and teacher referring to the mathematical content have been coded in the following way: The contributions are coded as 'object' (abbr. O), when they only contain aspects of a given problem-situation; when only aspects of the mathematical caculus or model are involved, the contributions are coded as 'sign' (abbr. S). Statements are interpreted as belonging to the level of 'concept', if they simultaneously contain elements of 'object' and of 'sign' in the shape of relations, and they are coded as 'relation' (abbr. R) accordingly. (A fourth category represented statements which could not be related to the other three, but referred indirectly to the mathematical content; all four categories were distinguished according to 'teacher explanation', 'teacher question' and 'student statement', giving a total of 12 different categories. In the following only the three "main" categories 'object' (O), 'relation' (R) and 'sign' (S) will be discussed.)

The basis of coding was an epistemological analysis of the mathematical tasks presented in the lessons which led to a differentiation between the level of 'object' and 'sign/model' with regard to the particular lesson. Two external coders performed the technical coding of the transcribed statements independently. The transcripts were divided into time intervals of 2 minutes before; statements were subdivided into semantic units - if necessary. Every semantic unit was coded according to the given coding schema; by means of a computer program, the lists of coded data were translated into graphical diagrams (see for instance A1): Every black beam represents a contribution (of the teacher or of a student) on the respective level of meaning; beams drawn through all three levels express the presentation of mathematical tasks. (For more details, particularly concerning the reliability of coding and the graphical representation of all 12 categories see Bromme/Steinbring 1987).

3. The mathematical topic of the lessons

The "complementarity of mathematical concepts" fundamentally inherent in probability as simultaneously empirical and theoretical conceptual aspects (probability as relative frequency and as relative portion) causes, even in the teaching of elementary stochastics, a distinction between simple models of probabilistic aspects (for instance in form of "ideal" random generators as the ideal coin or the ideal die, etc.) and intended real random phenomena (as for instance produced by games with dice or other random mechanisms). The complementaristic interplay between 'model' and 'situation' (cf. Dörfler 1986) is very typical for probability theory, but is basically a fundamental epistemological quality of every mathematical concept. It is the basis for characterizing mathematical knowledge as relations in the epistemological triangle, which serves as a conceptual means for coding the knowledge interactively negotiated in the classroom.

With regard to the four lessons concerned with elementary probability, the study describes how the meaning of knowledge develops in the interaction between teacher and students. The general topic of these four lessons is the introduction of the representational concept of "tree diagram" and its initial interpretative use. The situation used to begin the introduction for both teachers is a task describing a little boy who wants to go from his home to different playgrounds (soccer field, playground, swimming pool). At the crossings of his paths, the boy cannot decide which direction to follow, and he has the idea of leaving his choice entirely to chance by tossing a coin. If tails appear, he takes the path to the left, otherwise, he takes the path to the right. In this imagined real context, a tree diagram of two degrees must be elaborated as a "decision" diagram for analyzing this situation. The contrast between path-diagram and tree or decision-diagram expresses in an exemplary way the complementarity of representational and situational aspects of mathematical knowledge. Furthermore, the establishment of a relation between the path diagram and the decision diagram became a severe didactical problem which caused great difficulties of understanding for many students. In the lessons of teacher A, the tree diagram was treated and investigated in an experimental manner, i. e. by "simulating" the situation several times with coin experiments and by noting and discussing observed data. In the lessons of teacher B, the understanding of the tree diagram was mainly supported by some kind of terminological codification of paths and crossings, and by the construction of a schematized diagram (which should serve to count ideal numbers for determing the probabilities).

4. The graphical representations of the lessons

With regard to the differences between the lessons of teacher A and teacher B, the question is how the level of 'relationship' develops in the graphical lesson patterns. There, the differences in the graphical patterns become immediately salient. In teacher A's lessons, the relational level is of almost equal rank with the other two levels; for teacher B, however, this level has a subordinate position (cf. the diagrams A1, A2, B1, B2). This visual impression is created by the larger number of contributions and by the clearer structurization of the contributions of the middle - relational - level in case of teacher A. Teacher B, in contrast, treats this level far less than the other two (a fact proved by counting them out; 28%, resp. 37% of contributions on the relational level in the lessons of teacher A as opposed to 14% of contributions in the lessons of teacher B). Besides, it is seen that the relational level increases over time in case of teacher A. Teacher A's first lesson, in particular, clearly shows this gradual focussing on the relational level; in the second lesson, a homogeneously high proportion on the relational level is attained even earlier.

In contrast, the graphical lesson pattern of teacher B's lessons gives the impression that the relational level is never truly stabilized. During the first lesson (B1), the 'sign' resp. 'model' level predominates, while the level of 'object' seems to prevail as the second half of the lesson begins. In the second lesson (B2), it is evident that teaching switches back and forth between the 'object' level and the 'sign' level, and without any recognizable systematical integration of the relational level. Considering the graphical lesson patterns shows phenomenally that the two teachers handle the relational level quite differently.

5. The particular significance of graphical diagrams

Graphical representations of numerical data are not simply illustrative images offering a direct access to the data. Graphical diagrams must not be conceived of as imperfect pictures of teaching phenomena or other real situations which still have to be completed. They offer geometrical visual frames for exploring, explaining and analyzing hidden relations and structures in the data; graphical diagrams are theoretical means of exploration. "1. Graphical representations possess autonomous functions in processes of understanding, which in general cannot be substituted by other means. 2. Graphical representations are genuine cognitive means ... and do not belong only to the sphere of communication... 3. Graphical representations are explorative means. ... It is possible to operate with them formally relatively independent of references ... to contribute in this way to an investigation of unknown facts." (Biehler 1985, p. 70)

In the case of statistical data, this theoretical and explorative interpretation of graphical diagrams is particularly necessary. Here one has to take into account that the given set of statistical data is only a "representative" of a variety of "similarly" structured data. For discovering in the concrete and individual case of a fixed set of data, some general strucures and tendencies, graphical means of representation are extremely helpful if used in this theoretical and exploratory sense. The variation of graphical visualizations and the operation on graphical diagrams help discover general underlying structures which are inherent in the concrete individual case of observation.

The different exploratory functions of graphical lesson patterns, as developed for the analysis of knowledge development in the classroom, refer to different levels of investigation:

- global patterns of development and types of structures in the course of teaching
- local detailed structures and patterns of mutual effects of the interactions in the teaching process
- the separation of data types in contrasting patterns according to different characteristics of the real phenomena

6. An example: The separation of data types

The segregation of teacher's statements and also of students' statements is an important "graphical operation" belonging to the separation of data types into contrasting groupes in order to construct different graphical lesson patterns for the same lesson. The separations of the second lesson of teacher A (A2) and of teacher B (B2) as well, show in an exemplary way the particular significance of graphical representations (see the graphical lesson patterns A2T, A2S, B2T, B2S).

The separated graphical lesson patterns of teacher A's lesson (A2T, A2S) give the impression that the structure of knowledge development related to the teacher's statements is quite in conformity with the knowledge development caused by the students' statements; this means, the general pattern generated by all non-separated statements shows up more or less in each individual separated pattern. In contrast, the separation of the data of teacher B's lesson (B2T, B2S) leads to two differently structured graphical patterns. The pattern produced by the students' statements reinforces the graphical structure observable in the general pattern of this lesson. The switching back and forth between the 'object' level and the 'sign' level seems to be paricularly determined by the students', not by the teacher's contributions. The students' contributions dominate the particular structure of the graphical pattern, not the teacher's statements.

The separation of the contributions of students and of the teacher produces, for the lesson of teacher A, two graphical patterns with a similar structural course (also compared with the general graphical pattern of this lesson); for the lesson of teacher B, the separation produces two different graphical patterns among which the students' pattern reinforces the structure of the general pattern of this lesson.

The comparison of separated data with the help of different graphical lesson patterns exemplarily explains the theoretical peculiarity of graphical diagrams. On the one hand, graphical visualizations permit a concise representation of knowledge together with the possibility of discovering structures hidden in observed phenomena and situations. The quick overview of the whole structure of a lesson's course (by the general patterns and by the separated patterns as well) is an important possibility to comprehend a teaching lesson in a specific conceptual way, an achievement otherwise prevented by the great complexity of concrete teaching processes. This does not mean that graphical diagrams are simply reductions or incomplete models of real situations - in this respect, every theoretical concept must contain reductions (or abstractions) towards the complexity of concrete phenomena. It is important that graphical diagrams are theoretical means for recognizing new relations and developing a new theoretical perspective on seemingly known facts. With regard to this, the graphical lesson patterns offer a new conceptual view on the problem of the development of mathematical knowledge and its meaning in teaching/learning processes.

7. References

Bauersfeld H. (1982): Analysen zur Kommunikation im Mathematikunterricht, in: Bauersfeld et al. (ed.) Analysen zum Unterrichtshandeln, Köln, 1-40.

Biehler R. (1985): Graphische Darstellungen, in: math. did. 8, 57-81.

Bromme R./Steinbring H.(1987): Die epistemologische Struktur mathematischen Wissens im Unterrichtsprozeß. Eine empirische Analyse von vier Unterrichtsstunden in der Sekundarstufe I. Occ. Paper no. 90, Bielefeld

Dörfler W. (1986): Das Verhältnis mathematischer Operationen und gegenständlicher Handlungen, in: Steiner H. G. (ed.): Grundfragen der Entwicklung mathematischer Fähigkeiten, Köln, 1-14.

Mellin-Olsen S. (1987): The politics of mathematics education, Reidel.

Ogden C.K./Richards F.A. (1923): The meaning of meaning, London.

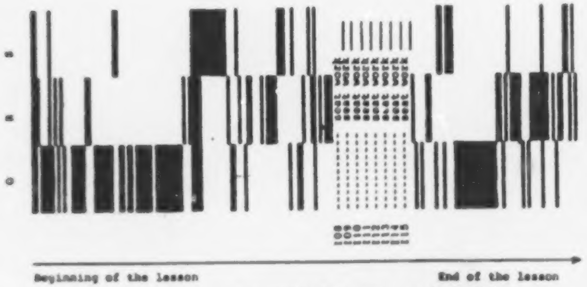
Otte M. (1984): Was ist Mathematik?, Occ. Paper no. 43, Bielefeld

Steinbring H. (1985): Mathematische Begriffe in didaktischen Situationen: Das Beispiel der Wahrscheinlichkeit, in: JMD, 6, 85-118.

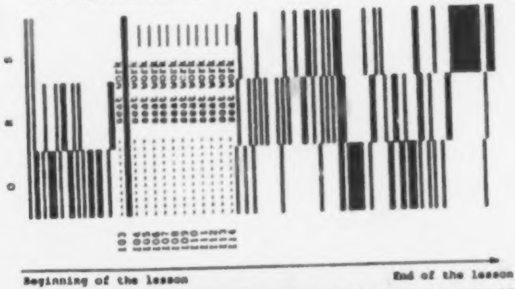
Steinbring H. (1988): Routine and meaning in the mathematics classroom, Paper presented to the third SCTP-Conference, London, April 10-15.

Voigt J. (1984): Interaktionsmuster und Routinen im Mathematikunterricht - Theoretische Grundlagen und mikroethnographische Falluntersuchungen. Weinheim, Beltz.

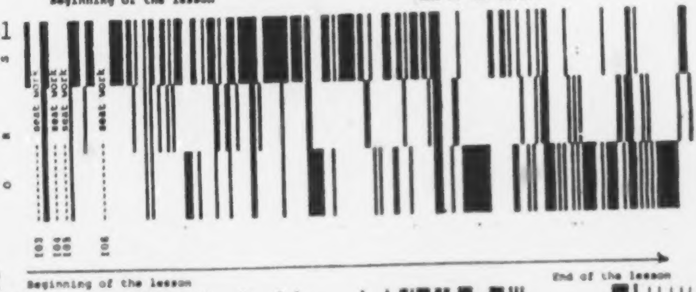
A1



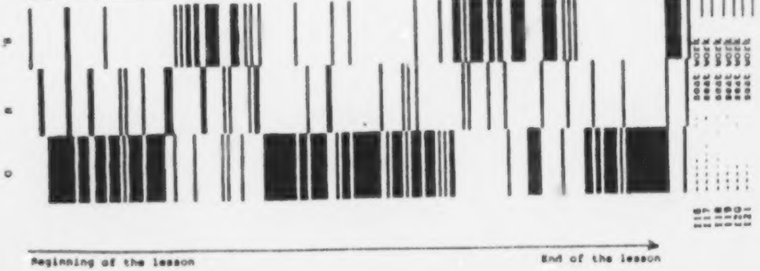
A2



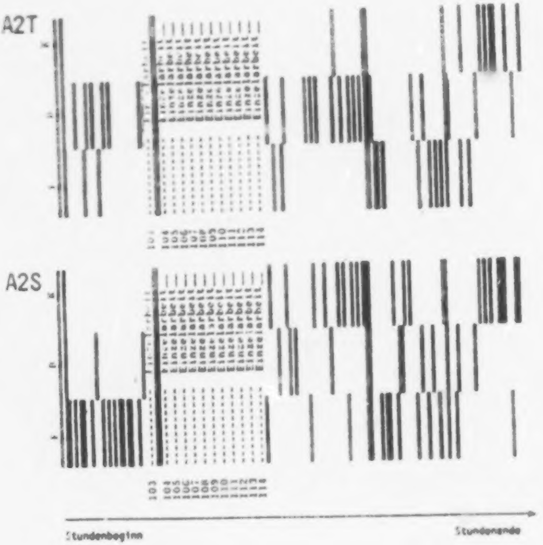
B1

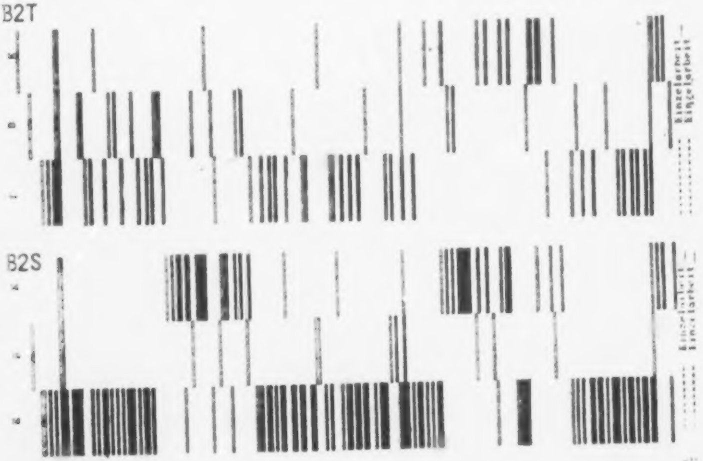


B2



272


A2T
103
104
105
106
107
108
109
110
111
112
113
114
Einzelarbeit
A2S
103
104
105
106
107
108
109
110
111
112
113
114
Einzelarbeit
Stundenbeginn
Stundenende


B2T
Einzelarbeit
Einzelarbeit
B2S
Einzelarbeit
Einzelarbeit
Stundenbeginn
Stundenende

LONGER-TERM CONCEPTUAL BENEFITS FROM USING A COMPUTER IN ALGEBRA TEACHING

Michael Thomas & David Tall

Mathematics Education Research Centre
University of Warwick, U.K.

This paper provides evidence for the longer-term conceptual benefits of a pre-formal algebra module involving directed computer programming, software and other practical activities designed to promote a dynamic view of algebra. The results of the experiments indicate the value of this approach in improving early learners' understanding of higher level algebraic concepts. Our hypothesis is that the improved conceptualisation of algebra resulting from the computer paradigm, with its emphasis on mental imagery and a global/holistic viewpoint, will lead to more versatile learning.

The Background

In a previous paper (Tall and Thomas, 1986) we described the value of a three week "dynamic algebra" module designed to help 11 and 12 year-old algebra novices improve their conceptual understanding of the use of letters in algebra. The activities include programming (in BASIC), coupled with games involving the physical storage of a number in a box drawn on card, marked with a letter, and software which enables mathematical formulae to be evaluated for given numerical values of the letters involved. This paper carries the work further with two experiments that test the nature of the learning and its longer term effects.

Theoretical Considerations

The formal approaches to the early learning of algebra have nearly always considered the topic as a logical and analytical activity with very little, if any, emphasis on the visual and holistic aspects of the subject. Many researchers, however, have identified the existence of two distinct learning strategies, described variously as serialist/analytic and global/holistic respectively. The essential characteristics distinguishing these two styles have been recorded (e.g. Bogen 1969), with the former seen as essentially an approach which breaks a task into parts which are then studied step-by-step, in isolation, whereas the latter strategy encourages an overall view which sees tasks as a whole and relates sub-tasks to each other and the whole (Brumby, 1982, p.244). Brumby's study suggests that only about 50% of pupils consistently use both strategies, thus meriting the description of versatile learners. The advantages of versatile thought in mathematics are described by Scott-Hodgetts :

> Versatile learners are more likely to be successful in mathematics at the higher levels where the ability to switch one's viewpoint of a problem from a local analytical one to a global one, in order to be able to place the details as part of a structured whole, is of vital importance. ...whilst holists are busy speculating about relationships, and discovering the connections between initially disjoint areas of mathematics, it may not even occur to serialists to begin to look for such links. [Scott-Hodgetts,1986, page 73]

These observations on learning styles correlate well with a number of physiological studies which indicate that the mind functions in two fundamentally different ways that are complementary but closely linked (see, for example, Sperry et al 1969, Sperry 1974, Popper & Eccles 1977). The model of the activity of the mind suggested by these studies is a unified system of two qualitatively different processors, linked by a rapid flow of data and controlled by a control unit. The one processor, the familiar one, is a sequential processor, considered to be located in the major, left hemisphere of the brain, responsible for logical, linguistic and mathematical activities. The other processor, in the minor, right hemisphere, is a fast parallel processor, responsible for visual and mental imagery, capable of simultaneously processing large quantities of data. The two processors are linked physically via the corpus collosum, and controlled by a unit located in the left hemisphere. This image of the two interlinked systems, one sequential, one parallel, is a powerful metaphor for different aspects of mathematical thinking. Those activities which encourage a global, integrative view of mathematics, may be considered to encourage the metaphorical right brain. Our aim is to integrate the work of the two processors, complementing logical, sequential deduction with an overall view, and we shall use the term *cognitive integration* to denote such an approach, with the production of a versatile learner as its goal (see Thomas 1988 for further details).

The approach to the curriculum described here uses software that is designed to aid the learner to develop in a versatile manner. In particular, the software provides an environment which has the potential to enable the user to grasp a gestalt for a whole concept at an intuitive level. It is designed to enable the user to manipulate *examples* of a specific mathematical concept or a related system of concepts. Such programs are called *generic organisers* (Tall, 1986). They are intended to aid the learner in the abstraction of the more general concept embodied by the examples, through being directed towards the generic properties of the examples and differentiating them from non-generic properties by considering non–examples. This abstraction is a *dynamic* process. Attributes of the concept are first seen *in a single exemplar*, the concept itself being successively expanded and refined by looking at a succession of exemplars.

The generic organiser in the algebra work is the "maths machine" which allows input of algebraic formulae in standard mathematical notation and evaluates the formulae for numerical values of the variables. The student may see examples of the notation in action, for example $2+3*4$ evaluates to $2+12=14$, and not to $5*4=20$`. Although this contravenes experience using a calculator, the program acts in a reasonable and predictable manner, making it possible to discuss the meaning of an expression such as $2+3a$ and to invite prediction of how it evaluates for a numerical value of a. In this way the pupils may gain a coherent concept image for the manner in which algebraic notation works.

The teacher is a vital agent in this process, acting as a mentor in guiding the pupils to see the generic properties of examples, demonstrating the use of the generic organiser, and encouraging the pupils to explore the software, both in a directed manner to gain insight into specific aspects, and also in free exploration to fill out their own personal conceptions. This mode of teaching is called the *enhanced Socratic Mode*. It is an extension of the Socratic mode where the teacher discusses ideas with the pupil and draws out the pupil's conceptions (Tall, 1986). Unlike the original Socratic dialogue, however, the teacher does not simply elicit confirming responses from the pupil. After leading a discussion on the new ideas to point the pupils towards the salient features, the teacher then encourages the pupils to use exactly the same software for their own investigations.

The generic organiser provides an external representation of the abstract mathematical concepts which acts in a cybernetic manner, responding in a pre-programmed way to any input by the user, enabling both teacher and pupil to conjecture what will happen if a certain sequence of operations is set in motion, and then to carry out the sequence to see if the prediction is correct.

The computer provides an ideal medium for manipulating visual images, acting as a model for the mental manipulation of mathematical concepts necessary for versatility. Traditional approaches which start with paper and pencil exercises in manipulating symbols can lead to a narrow symbolic interpretation. Generic organisers on the computer offer anchoring concepts on which concepts of higher order may be built, enabling them to be manipulated mentally in a powerful manner. They can also encourage the development of holistic thinking patterns, with links to sequential, deductive thinking, which may be of benefit in leading to better overall performance in mathematics.

Longer-Term Benefits in Algebra

In order to test the long-term effects of the "dynamic algebra module", a follow-up study was carried out over one year after the initial experiment previously described (Thomas and Tall, 1986). By this time the children were now 13 years old and had transferred to other schools where they had completed a year of secondary education. Eleven of the matched pairs attended the same secondary school and were put into corresponding mathematics sets, so that during their first year (aged 12/13) they received equivalent teaching in algebra. At the end of the year they were all given the algebra test used in the original study. A summary of the results and a comparison with their previous results are given in table 1. This demonstrates that, more than one year after their work on basic concepts of algebra in a computer environment, they were still performing significantly better.

Test	Experim. Mean (max=79)	Control Mean (max =79)	Mean Diff.	S.D.	N	t	df	p
Post test	32.55	19.98	12.57	10.61	21	5.30	20	<0.0005
Delayed Post-test	34.70	25.73	8.47	11.81	20	3.13	19	<0.005
one year later	44.10	37.40	6.70	7.76	10	2.59	9	<0.025

Table 1

This lends strong support to the idea that the introduction of a module of work, such as the dynamic algebra package, with its emphasis on conceptualization and use of mental images rather than skill acquisition, can provide significant long-term conceptual benefits.

Skills and Higher Order Concepts

A second teaching experiment was held in which a dynamic algebra approach using the computer was compared with more traditional teaching methods. The subjects of this second experiment were 12/13 year old children in six mixed ability classes in the first year of a 12-plus entry comprehensive school. The school is divided into two halls with children apportioned to provide identical profiles of pupil ability, but the teaching is done by a unified team of teachers, allowing direct comparisons of different teaching methods. On the basis of an algebra pre-test it was possible to organise 57 matched pairs covering the full ability range in the classes.

In the first stage [illegible] the comparison the experimental group used the dynamic algebra module during their normal mathematics periods, using computers in small groups of two or three over a three week period during the autumn term. At the same time the control group used a traditional skill-based module employed in the school over some years, covering basic simplification of expressions and elementary equation solving in one unknown. Immediately following the work they were given a post-test containing the same questions as the pre-test.

The results (given later in table 5) superficially showed that there was no significant difference in overall performance, but analysis of individual questions presented an interesting picture. On skill-based questions related to the content of the traditional module, the control group performed significantly better, whilst on questions traditionally considered to be conceptually more demanding, the experimental group performed better. Table 2 shows typical skill-based questions and the better performance of the control group:

Question	Experimental %	Control %	z	p
Multiply 3c by 5	14	41	3.07	<0.005
Simplify 3a+4b+2a	50	73	2.46	<0.01
Simplify 3b-b+2a	29	61	3.36	<0.0005
Simplify 3a+4+a	38	78	1.60	n.s.
G jigsaws and J jigsaws =?	55	78	2.39	<0.01

Table 2

Table 3 shows the better performance of the experimental group on questions considered to be more demanding in a traditional approach, requiring a higher level of understanding, including the concept of a letter as a generalized number or variable:

Question	Experimental %	Control %	z	p
For what values of a is a+3>7 ?	31	12	2.33	<0.01
For what values of a is 6 > a+3 ?	22	6	2.33	<0.01
a+b=b, always, never, sometimes ... when?	31	17	1.65	<0.05
M+P+N=N+M+R, always, never, sometimes ... when?	38	28	1.08	n.s.
Perimeter of rectangle D by 4	50	27	2.46	<0.025
Perimeter of rectangle 5 by F	50	29	2.24	<0.025
Larger of 2n and n + 2 ?	7	0	1.91	<0.05

Table 3

The differential effects of the two treatments could be considered as a manifestation of the skills versus conceptual understanding dichotomy, in terms of the levels of understanding defined by Küchemann [1981]. His level 1 involves purely numerical skills or simple structures using letters as objects, level 2 involves items of increased complexity but not letters as specific unknowns. Level 3 requires an understanding of letters as specific unknowns; level 4 requires an understanding of letters as generalized numbers or variables. It is important to understood that these levels were not intended to be a hierarchy but rather a description of children's functional ability. However, it is only at levels 3 and 4 that children are really involved in algebraic thinking rather than arithmetic and few children (17% at age 13) attain this level of understanding. Table 2 shows that the control pupils outperform the experimental pupils at levels 1 and 2, whilst table 3 shows that the experimental pupils outperform the control pupils at the higher levels.

This suggests that there are differential effects from the two approaches in respect of surface algebraic skills (in which the control students have a greater facility at this stage) and deeper conceptual understanding (in which the experimental students perform better). An alternative (and, we suggest, more viable) explanation is that the traditional levels of difficulty depend on the approach to the curriculum and may be altered by a new approach using the computer to encourage versatile learning.

Longer-term effects on skills and conceptual ideas

In the summer term, some sixth months later, the pupils were all given the same traditional revision course on their earlier algebra, without any use of the computer. Both groups were re-tested and a comparison of matched pairs was made again. Table 4 shows the pupils performance on the test as a whole. On this occasion the experimental students now performed significantly better than the control students.

Test	Experim. Mean (max=67)	Control Mean (max=67)	Mean Diff.	S.D.	N	t	df	p (1-tail)
Post test	36.0	35.9	0.1	10.46	47	0.06	46	n.s.
Delayed Post-test	42.1	39.3	2.76	8.91	46	2.08	45	<0.025

Table 4

In the conceptually demanding questions of the type mentioned in table 1, the experimental students continued to maintain their overall superiority (table 5).

279

Test	Experim. Mean (max=26)	Control Mean (max=26)	Mean Diff.	S.D	N	t	df	p (1-tail)
Post test	9.28	7.77	1.50	4.81	48	2.14	47	<0.025
Delayed Post-test	10.97	9.45	1.51	4.73	47	2.17	46	<0.025

Table 5

Meanwhile, on the skill-based questions, the experimental students marginally surpassed the control students, although the difference was not statistically significant.

The effects of Gender

Although the researchers did not set out to look specifically at the relationship between performance and gender, a factor analysis including ability and gender among its variables was included. A random sample of girls and boys was taken and a comparison on pre-test and post-test made. In the sample the girls performed less well than the boys on the pre-test, but made a statistically significant improvement to perform better than the boys on the post-test. The reasons for this are not altogether clear at this stage. It was certainly noticeable that the more able boys, with previous computer experience, were constantly showing their prowess at making the computer print screensful of coloured characters, and some saw the elementary activities as a little beneath their dignity. Meanwhile some of the girls had initial difficulties and took the task extremely seriously, discussing the problem and helping each other in small groups. Thus the experiment was unable to distinguish whether the difference was social or cognitive.

Conclusions

The experiments provide evidence of a more versatile form of thinking related to the computer experiences. Further this improved understanding of concepts usually considered to be of a higher level and difficult to attain by traditional methods, was shown to of a long-term nature. There is also support for the hypothesis that the computer can be used in the enhanced Socratic mode to provide experiences to encourage versatile learning through cognitive integration.

References

Bogen J.E. 1969: 'The Other side of the Brain : 2. An Appositional Mind', *Bulletin of the Los Angeles Neurological Society*, 34, pp. 135-162.

Brumby M.N. 1982: 'Consistent differences in Cognitive Styles Shown for Qualitative Biological Problem Solving', *British Journal of Educational Psychology*, 52, 244-257.

Küchemann D.E. 1981: 'Algebra', in Hart K.M. (ed.),1986, pages 102-119.

Popper K.R. & Eccles J.C. 1977: *The Self and its Brain*, Springer, Berlin, 1977.

Scott-Hodgetts R. 1986: 'Girls and Mathematics: The Negative Implications of Success', in *Girls into Mathematics Can Go*, Burton L. (ed.), Rinehart & Wilson, London, 61-76.

Sperry R.W., Gazzaniga M.S. & Bogen J.E. 1969: 'Interhemispheric Relationships: The Neocortical Commissures: Syndromes of Hemispheric Deconnection', in Vinken P.J. & Bruyn G.W. (eds *Handbook of Clinical Neurology*, Vol. 4, North Holland Publishing Co., Amsterdam, 273-290.

Sperry R.W. 1974:Lateral Specialisation of Surgically Separated Hemispheres, in Schmitt F.O. & Worden F.G. [eds.], The Neurosciences Third Study Program, MIT Press, Mass., 5-19.

Tall D.O. 1986: *Building and Testing a Cognitive Approach to the Calculus Using Interactive Computer Graphics*, Ph.D. Thesis, The University of Warwick.

Tall D.O. & Thomas M.O.J. 1986: 'The Value of the Computer in Learning Algebra Concepts', *Proceedings of the 10th Conference of P.M.E.*, London.

Thomas M.O.J. 1988: *A Conceptual Approach to the Early Learning of Algebra Using a Computer*, unpublished Ph.D. Thesis, The University of Warwick, (in preparation).

THE ROLE OF AUDIOVISUALS IN MATHEMATICS TEACHING

TOMPA, KLÁRA: National Centre for Educational Technology
Hungary

The use of a number of means characterises the mathematics classes. Objects which can be taken into one's hand or their pictures can greatly promote the active participation of children in the process of problem solving and concept formation. The functions of the symbols as defined by R.R. Skemp are very well realised by means of the audio-visual media. Explanation, understanding, the promotion of the abstraction process and other 'symbol-functions' can be realised with the help of the representational possibilities of slides, overhead transparencies, films and videos.

The teaching of mathematics has greatly changed over the past decades. The basic reason for the change was in mathematics itself, in the development of mathematics. The development of the discipline of mathematics, its self-renewal made it so effective that more and more other fields of discipline apply mathematics in ever newer ways /economics, linguistics, psychology, computer science, etc./. Application means two things: the application of new fields of mathematics on the one hand, and the application of a mathematical attitude, way of thinking, activity on the other. Nowadays all professions require specialists who are familiar with the methods and attitude of mathematics, what's more, who are able to apply mathematics as well. So the most important task of the teaching of mathematics beside providing a certain amount of factual knowledge, is also the shaping of the personality with the help of mathematics. In order that mathematics should be built into the personality as a way of thinking and form of activity the pupil has to face a large number of situations in which they can trace the feel of mathematical kinds of activity, try them out themselves, practise them and on the basis of several individual cases they can formulate their characteristic features. So the process of the teaching and learning of mathematics has changed considerably in a number of ways:

- In its contents and structure.
- Pupils do not only use paper and pencil in a class of mathematics but a number of other things /especially in the lower elementary forms/.Mathematical activities needed for concept formation and problem solving are carried out by the children with the help of objects taken into their hand or their picture symbols.
- The behaviour of pupils during class has also changed. Instead of being a passive receiver he is now an active participant, not only in his psycho-motoric manifestations but in the field of cognitive processes as well.
- The teachers' behaviour in preparation for, and during the class has changed as well, it has become richer. Beside the offering of information new tasks are set for the teacher, like the organisation of the work of individual pupils and small groups as well as the direction of this work, giving extra jobs to the very fast ones and the very slow ones, etc.

This changed process of teaching and learning requires the more unified design of the contents, the means, the methods, the different activities of the teacher and the pupil.
The present paper deals only with one of the means of mobilising the pupils for active participation, i.e. audio-visual aids and the opportunities offered by them.

What can audio-visual aids offer for the teaching of mathematics?

Because of the nature of the subject first of all the audiovisual media , like slides, overhead transparencies, films and videos can be used in the teaching of mathematics. /I do not wish to deal with the ever growing role of the computer and the extremely useful possibility of interactivity./

We can have great expectations when we use visual aids because mathematics uses visual symbols very comprehensively. R.R. Skemp differentiates between ten different functions of symbols, which are: communication, restration of information, formation of new concepts, the facilitating of multiple classification, explanation, understanding, the facilitating of the representation of structures, the formation of routine skills, the recollection and the understanding of bits of information. /Skemp, 1975/

The above mentioned audio-visual media can very well realise these functions of symbols.

Let me show one by one what each of the visual media is capable of doing in promotion of the teaching of mathematics.

SLIDES, SLIDE SERIES

Slides might be very different as far as their representation technique is concerned, ranging from true, realistic coloured pictures of objects to simple linear drawings showing the outline of objects.

Looking at it from a different point: graphic slides prepared with a clear representation method and showing aesthetic qualities as well may greatly help convey information about things which cannot be sensed by vision. Mathematics teaching can make best use of such colourful graphic slides.They are the ones which can make the visual symbols indispensable in various fields of mathematics much more efficient.

We have prepared 405 coloured graphic slides to be used in the mathematics classes of the lower primary grades /aged 6 to 10/ with the following expectations:

- The redundance of lengthy explanations and information giving can be avoided or decreased with the help of the adequate combination of mathematical and graphic symbols.
- The ready made slides which can be projected on the spotsave

the energy and time of the teacher, which he can spend in some useful other way.

- As the majority of the slides contain problems and tasks to be solved, they can be used as "a visual collection of problems" to be used for teaching any new area of the teaching material.
- The representation of the tasks individual use possible. this way, using individual slide-viewers the pupils can be given individualised tasks.
- The simple, aesthetic pictures in line with the taste of the lower elementary age group help the pupils carry out real mathematical activity. The graphic representations might also help visual training beside the teaching of mathematics.

Children generally like film projections and working with slide viewers. All the same the application of these slides is only effective if the pupils do the tasks and mathematical activities that the slides tell them to. And as the tasks require serious work the teacher has to decide very carefully which slide to use, when and who to give it to.

OVERHEAD TRANSPARENCIES

Overhead projectors are the most widespread aids used in the most various ways in Hungarian classrooms nowadays. This is understandable because there are so many kinds of transparencies possible. The one consisting of one page or several pages building up the figure or the ones that can be moved can all fulfil a number of functions in the process of teaching in general and also in teaching mathematics.

- Among the "one-page" transparencies great importance can be given to the ones which can be used to help work in the class. They usually contain some basic figure, network and the teacher and the pupils working together or separately prepare on it some more complicated figure, having important details, showing important connections, relationships. Typical examples are square grids, number lines with different units, systems of axes, empty charts, Venn diagrams, auxiliary grids and other.

- The transparencies consisting of several pages, the so called building-up transparencies can also be used for multiple functions. They can help prove mathematical theorems step by step. The given bits of information may guide or promote the thinking towards the possible solutions. In problem solving the pupils can check their solution by turning the page with the solution onto the original task page.
- Common problem solving can also be helped by the teacher with the help of figures built up uf several steps.
- Overhead transparencies containing movable parts can also be of great use in the mathematics class. E.g. the understanding of geometrical transformations, function transformations can be made much clearer, much easier to understand.

FILMS, VIDEO RECORDINGS

Films and videos as other audio-virual media have proved that they are capable of transferring true knowledge and thus widening the range of experiences of pupils. They do so because they are capable of the following:

- They can show processes which cannot be viewed in any other way.
- They can widen the limits of human perception. With the technique of speeding up or slowing down they might show processes to the viewers which would otherwise be not perceptible for the human eye. The possibility of reducing and amplifying and other special techniques all open up the limits of the human eye and observation so as to be able to see phenomena not perceived earlier.

- Limitations of space and time can be overcome with the help of films and videos. A possibility is opened up for the recording and multiple viewing of rare phenomena which can thus be shared with others.
- The application of animation and computer graphics in films and videos makes it possible to picture things and phenomena not perceivable visually.
- Film and video give the feeling of involvement and the experience of presence more than any other programs using pictures for illustration.

Taking into account all these characteristic features of films and videos it can be said that motion pictures on mathematical topics /especially from the field of geometry/ use animation and graphic techniques, because mathematics operates with symbols on the level of abstraction first of all, although its concepts are rooted in reality. With the motion of plain and spatial figures hardly conceivable facts can be made visible like the one that there is no shortest one among the chords of a circle but there is a longest one.

Films are generally not prepared for individual learning, but they are shown to the whole class. Their viewing must always be prepared with great care, so that the pupils were able to perform the mental processes the film asks them to during the viewing.

Slides, transparencies, films and videos can be used right in the class thus promoting the teaching of mathematics. But on the other hand there are means which exert their influence in an indirect way, like the mass media, radio and television which might broadcast mathematical tasks and problems for the children interested in the form of a competition. Videos may have other impacts on mthematics teaching. In teacher training students' micro-teaching can be recorded thus developing the efficiency of their own teaching. The efficiency of the teaching and the performance of the students are greatly influenced by the teacher's ability of setting the problems, his skill of asking questions.

HOW TO EDIT THE INFORMATION

It must be clear for those who develop audio-visual media that learning from a picture is a different process from learning from a book by reading. Reading is a linear process of putting the words one after the other, building up the meaning of a sentence. As opposed to this the information contained in a picture is present at the same time and it depends on the viewer how, in what order he comprehends the information gained from each part. Besides, the motion picture might disappear too fast, before its essence could be understood. Taking all this into account visual information must be edited very carefully so that it had the desired result in learning. There are some practical bits of advice to be followed during the technical realisation of the pictures:

- Text and figure should support cr complete each other but they should never repeat what the other sais.
- Visual elements and inscriptions shoud be clearly organised so as to convey an aesthetis message as well.
- Irrelevant details should be omitted, pictures should not be overcrowded.
- Types and sizes of letters and numbers should be carefully conceived and not varied too often.
- In order to avoid false impressions the pictures should contain some points of reference about the size of figures.
- Taking into consideration one of the important elements of human learning, i.e. selective perception all must be done to direct the attention to the important features.+

We have to underline the most important details./Arrows, coloured plots, numbering, lettering, frames, animation, repetition, slowing down, speeding up, electronic light effects, etc./.

Carefully edited witty audio-visual **media** can raise the attention and keep it awake, so they are very effective aids in the teaching of mathematics.

+ /Gagné, 1980/

LITERATURE

1. R.R. Skemp, A matematika tanitás pszichológiája, Budapest 1975. Hungarian translation of The Psychology of Learning Mathematics, Penguin Books, Harmondsworth, 1971
2. R.M. Gagné,"Is Educational Technology in Phase?",Ed. Techn. 1980. 2. pp. 7-14

SPECIFYING THE MULTIPLIER EFFECT ON CHILDREN'S SOLUTIONS OF SIMPLE MULTIPLICATION WORD PROBLEMS

L. Verschaffel, E. De Corte & V. Van Coillie
Center for Instructional Psychology
University of Leuven, Belgium

Abstract

One important finding from recent research on multiplication word problems is that children's performances are strongly affected by the nature of the multiplier (whether it is an integer, decimal larger than 1 or a decimal smaller than 1). On the other hand, the size of the multiplicand has little or no effect of problem difficulty. The aim of the present study was to collect empirical data concerning this "type of multiplier" effect in combination with two additional task variables which have not yet been seriously addressed in previous research, namely (1) the symmetrical/asymmetrical character of the problem structure and (2) the mode of response (choice of operation versus free response mode). While the data of the present study provide additional evidence for the above-mentioned effect-of-multiplier hypothesis, they also show that the two other task variables also strongly influence children's difficulties with multiplication problems.

INTRODUCTION

During the last years researchers have started to analyze pupils' solution skills and processes with respect to multiplicative word problems (for an overview see Bell, Grimison, Greer & Mangan, 1987). A robust finding from these studies is that children's difficulty in choosing the correct operation depends strongly on the nature of the multiplier. For example, Mangan (1986) found that children performed significantly better on problems with an integer as multiplier than when the multiplier was a decimal larger than 1; problems with a multiplier smaller than 1 were still much more difficult. (The most common error on the latter problem type was dividing instead of multiplying the two given numbers.) On the other hand, the size of the multiplicand had no significant effect on problem difficulty.

Fischbein, Deri, Nello and Marino (1985) have developed the following theoretical account for these findings: each arithmetical operation remains linked to an implicit and primitive "intuitive model", which mediates the identification of the arithmetic operation needed to solve a word problem. According to the authors, the primitive model associated with multiplication is "repeated addition", in which a number of collections of the same size are put together. A first consequence of this "repeated addition" model is that, while the multiplicand can be any positive number, the multiplier must be an integer. A second implication is that multiplication necessarily results in a number that is bigger than the multiplicand. When these constraints of the underlying model are incongruent with the numerical data given in the problem, the choice of an inadequate operation may be the result (Fischbein et al., 1985). While the available experimental and observational data concerning the effect of number type are consistent with Fischbein et al.'s (1985) theory, there still remain several questions requiring further investigation.

First - with the exception of Mangan's recent study (1986) - the evidence on the effects of the type of multiplier on the choice of operation (regardless the nature of the multiplicand) is not convincing, because it is based on comparisons between problems that differ also in several aspects other than the nature of the numbers (Bell et al., 1987). Consequently, a first objective of the present study was to collect additional data about the effects of type of multiplier and type of multiplicand in a more carefully designed way.

Second, the word problems included in previous investigations always had asymmetrical structures. This means that the two quantities multiplied play psychologically a different role in the problem situation, and are therefore non-interchangeable. This raises the question whether the type of the given numbers affects also the solution of symmetrical problems, in which the roles played by the quantities multiplied are essentially equivalent.

Third, in most previous studies pupils were not asked to answer the problems, but to indicate which formal arithmetic operation would yield the correct solution. However, selecting a formal arithmetic operation with the two given numbers, is not the only way in which a one-step word problem can be solved. Besides, there are a lot of informal solution strategies that may lead to the correct answer. Therefore, one could ask whether the number of wrong-operation errors would be as large when the item format does not force children to choose a formal arithmetic operation, but allows them to rely on other, more informal solution strategies.

METHOD

A paper-and-pencil test consisting of 24 one-step problems was constructed. The test contained 16 multiplication problems; the remaining eight items were included to reduce the likelihood of stereotyped, mindless response strategies on the 16 target problems. Half of the multiplication problems had an asymmetrical structure (rate problems like "One litre of milk costs x francs; someone buys y litres; how much does he have to pay?"); the other half were symmetrical (area problems like "If the length is x meters and the breadth is y meters, what is the area?"). All eight symmetrical and asymmetrical problems differed with respect to the type of the multiplier or the multiplican (either an integer, a decimal larger than 1, or a decimal smaller than 1). This 24-items test was given to a group of 116 sixth-graders twice: once in a choice-of-operation form and once in a free-response form.

Afterwards an analysis of variance (with a randomized block factorial design) was performed with the following four task characteristics as independent variables: (1) type of multiplier: an integer, a decimal larger than 1 or a decimal smaller than 1; (2) type of multiplicand: an integer, a decimal larger than 1 or a decimal smaller than 1; (3) problem structure: symmetrical or asymmetrical; (4) response mode: choice of operation or free response. In the multiple-choice format, the dependent variable was the number of children that indicated the correct operation; in the free-response format it was the sum of the correct answers and the technical (or computational) errors, the underlying idea being that answers resulting in technical errors nevertheless reflect correct thinking about the problem as is shown by the appropriate solution strategy chosen. Main and interaction effects significant at the 5 % level were further analyzed using Duncan's multiple range test ($p < .05$).

RESULTS

Main effects

The results of the analysis of variance revealed a significant main effect for the independent variable type of multiplier (F (2,3565) = 237.75, $p < .001$). A supplemental analysis (using Duncan's test) showed that the problems with an integer as multiplier were significantly easier than those where the multiplier is a decimal larger than 1, and that the latter were easier than problems with a multiplier smaller than 1. The proportion of appropriate solution strategies for these three problem types was .94, .89 and .71 respectively. On the contrary, no main effect was found for the independent variable type of multiplicand: the proportion of correct strategy choices for multiplicand as integer, decimal larger than 1, and decimal smaller than 1 was .86, .83 and .82 respectively. To summarize, our results confirm the hypothesis that the type of multiplier strongly influences children's choice of an appropriate solution strategy, while the nature of the multiplicand has no significant effect on their choices.

The analysis of variance also showed a main effect for the third independent variable, namely problem structure (F (1,3565) = 55.69, $p < .001$). The supplemental test revealed that the symmetrical problems elicited a larger proportion of correct strategies (.88) than the asymmetrical ones (.80). However, in this study symmetrical as well as asymmetrical problems were represented only by one single subtype (respectively "rate" and "area"). It therefore would be premature to conclude that in general symmetrical problems are easier than asymmetrical ones.

Finally, there was no significant difference between the proportion of correct operations for the problems presented in the two response modes, namely multiple choice (.83) and free response (.85).

Interaction effects

A main goal of the present study was to analyze how two additional task characteristics, namely problem structure and response mode, affect the influence of the type of multiplier on the proportion of correct strategy choices.

First, the analysis of variance showed a significant disordinal type of multiplier by problem structure interaction (F (2,3565) = 295.72, $p <$

.001). The supplemental Duncan test, based on $p < .05$, revealed that for the asymmetrical structure, problems with an integer as multiplier were significantly easier than those with a decimal larger than 1 as multiplier, and that the latter were easier than those in which the multiplier was a decimal smaller than 1 (see Table 1); this is entirely in line with the overall results reported in the previous section. For the symmetrical structures, on the other hand, there was much less difference between the proportions of correct strategy choices for the three distinct "type of multiplier" problems. Moreover, although here too significant differences were found, they were not in the expected direction: "decimal smaller than 1" and "integer" problems were both significantly easier than "decimal larger than 1" problems, but did not differ mutually (see Table 1). Furthermore, a comparison between the proportion of correct operations in the context of a symmetrical and asymmetrical structure for each of the three types of multiplier, revealed that integer and decimal larger than 1 problems were easier when embedded in an asymmetrical structure; for problems with a decimal smaller than 1, on the other hand, the symmetrical structure was the easiest. All three differences were significant.

Table 1. Proportion of appropriate solution strategies for the distinct "type of multiplier" problems in the two problem structures

Type of multiplier	Problem structure	
	Asymmetrical	Symmetrical
Integer	.99	.89
Decimal larger than 1	.93	.84
Decimal smaller than 1	.52	.90

A significant disordinal type of multiplier by response mode interaction was also found ($F(2,3565) = 53.28$, $p < .001$). The Duncan test revealed that in both response modes, problems with an integer as multiplier were significantly easier than those with a decimal multiplier larger than 1, and that the latter were in turn significantly easier than those having a decimal smaller than 1 as multiplier (see Table 2). However, when we compared the proportion of correct operations in both response modes for each of these three types of multiplier, it

was observed that "integer" and "decimal larger than 1" problems were easier in the choice-of-operation than in the free-response condition, whilst the reverse was true for problems having a decimal smaller than 1 as the multiplier. All three differences were significant.

Table 2. Proportion of appropriate solution strategies for the distinct "type of multiplier" problems in the two response modes

Type of multiplier	Response mode	
	Choice of operation	Free response
Integer	.97	.91
Decimal larger than 1	.91	.86
Decimal smaller than 1	.64	.78

DISCUSSION

Recent research on multiplication word problems has shown that problems with an integer as multiplier are much easier than those with a decimal multiplier larger than 1, and that problems with a multiplier smaller than 1 are still more difficult. By contrast, the nature of the multiplicand seemed to have only a marginal effect on problem difficulty. Generally speaking, the results of the present study support these findings. However, our results enable us to specify the "multiplier effect hypothesis" in two respects: (1) the differential effect of number type for the multiplier is only found in asymmetrical problems, not in symmetrical ones, and (2) this differential effect is much weaker in a free-response situation as compared to a forced-choice format.

The observed multiplier by problem structure interaction raises an important question, namely what mechanisms might account for the absence of a "type of multiplier" effect in our symmetric problems. In line with with Fischbein et al.'s (1985) theory, one could argue that the constraints of the "repeated addition" model do not affect negatively the solution process of symmetrical problems with decimals, because their symmetry does not require the problem solver to attribute the role of multiplicand and multiplier to particular numbers. But another explanation might be that the representation of "area" problems is not influenced by the "repeated addition" model, but rather by another primitive model, such as the "rectangular pattern" model (with other constraints imposed on the numbers that can be used and their role in the structure of the problem). A final plausible account for the absence of the multiplier effect in our symmetrical problems is that pupils' selection of the operation does not result from a mindful matching of the "deep" understanding of the problem structure with a formal arithmetical operation (mediated by a primitive model), but is simply based on the direct and rather mindless application of a well-known formula (area = length x breadth), associated with the key word "area" in the problem text.

The multiplier-response mode interaction is the second additional finding of our study: the negative influence of the multiplier being a decimal smaller than 1 was much weaker in the free-response than in the multiple-choice format. Our collective paper-and-pencil tests did not yield much information about the precise nature of the cognitive processes in the free-response mode that led to the correct strategy choice on problems with a multiplier smaller than 1. Previous work has demonstrated that pupils can often solve correctly simple multiplication problems with small integers using informal strategies without apparently being aware that the solution could be obtained by multiplying the two given numbers. However, the specific question raised by our data is: which solution paths - other than multiplying the two given numbers - can lead to the solution of a problem in which the multiplier is a decimal smaller than 1?

In view of answering the remaining questions we intend to collect in our future work more systematically data on children's solution processes while solving different types of multiplication problems using individual interviews and eye-movement registration as the main data-gathering techniques.

To conclude, whilst our data about the interaction effects of type of multiplier with problem structure and response mode are not necessarily inconsistent with Fischbein et al.'s (1985) theory, they suggest nevertheless that we may have to search for a more detailed and more comprehensive theory, based on the principle that the selection of an appropriate solution strategy is affected by a large number of factors competing for attention and interacting in complex ways (see also Bell et al., 1987).

REFERENCES

Bell, A., Grimison, L., Greer, B., & Mangan, C. (1987). Multiplicative word problems. A classification scheme and its validation. (Internal report). Nottingham/Belfast: Shell Centre for Mathematics Education, University of Nottingham/Department of Psychology, Queen's University.

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. Journal for Research in Mathematics Education, 16, 3-17.

Mangan, C. (1986). Choice of operation in multiplication and division word problems. (Unpublished doctoral dissertation). Belfast: Queen's University.

Is There Any Relation between Division and Multiplication? Elementary Teachers' Ideas about Division

Shlomo Vinner & Liora Linchevski
Israel Science Teaching Center
Hebrew University, Jerusalem

Abstract

Some aspects of division with whole numbers and with fractions were examined in 300 elementary teachers and preservice teachers. One of the main questions was whether these teachers have any kind of formal approach to division or they only have concrete models for it as the quotative and the partitive divisions. We also tried to expose these models by direct methods and not by indirect methods used in previous studies (Fischbein et al., 1985 and others). In addition to well known results as "multiplication makes bigger and division makes smaller" we also found the beliefs that multiplication by a fraction makes smaller and division by a fraction makes bigger. About 64% failed to point at the relation between division and multiplication when asked about in a particular question.

Several studies have been done on children's ideas about multiplication and division (Bell et al., 1981; Hart, 1981; Fischbein et al., 1985) and also on preservice teachers (Tirosh et al., 1986; Tirosh et al., 1987). The hypothesis was that certain models for the multiplication and division imply certain ideas about these operations.

The aim of this study is to extend the above studies in two dimensions. 1. We try to investigate the models for division and the ideas about multiplication and division directly and not in an indirect way as in the above studies. This we do by asking questions that stimulate the respondents to speak directly about their models and ideas. 2. About 3/4 of our study population were inservice elementary teachers and about 1/4 were preservice elementary teachers whereas the former studies examined either children (Bell et al., 1981; Fischbein et al., 1985; and Hart, 1981) or preservice elementary teachers (Tirosh et al., 1986, 1987).

Together with the view that multiplication makes bigger there exists a belief that multiplying by a fraction makes smaller. We examined how common this view is in teachers. We dealt also with the problem of division by zero as part of the models for division. We assumed that this problem could help us to determine whether the teachers have concrete models for division or formal models, as implicitly assumed by Fischbein (1985). The problem of division by zero is connected with the problem of the relation between multiplication and division, a problem with which we also deal in our study.

Method

Questionnaire

In order to create a questionnaire we interviewed several teachers. The interviews led us to form the following questions:

1. In an in-service teacher training course, the following question was posed to the supervisor: Is it possible to explain division in a mathematical way without telling stories about dividing cakes to children or similar stories? For instance, what is 381 : 84 ? What would you tell this teacher if you were the supervisor?
2. The operation 15 : 3 or even 3 : 15 can be explained by cakes divided to children. Does the operation $\frac{4}{2} : \frac{4}{3}$ have a similar meaning or is it only a formal operation?
3. Given 18 : 3, is this a partitive division or a quotative division?
4. How much is 5 : 0 ? Please, explain your answer!
5. A student claimed that any number divided by itself makes 1. Therefore, also 0 : 0 = 1. What is your reaction?
6. Which of the following is the most suitable for demonstrating that $4\frac{1}{2} : 3 = 1\frac{1}{2}$?

   (a) $3 \times 1\frac{1}{2} = 4\frac{1}{2}$

   (b) $9 : 4\frac{1}{2} = 2$

   (c) $1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2} = 4\frac{1}{2}$

   (d) $4\frac{1}{2} - 1\frac{1}{2} - 1\frac{1}{2} - 1\frac{1}{2} = 0$

   (e) $1\frac{1}{2} \times 3 = 4\frac{1}{2}$
7. In each of the following pairs of excercises, circle the one which gives a greater result. Please, explain your answer!

   I (a) 8 × 4 (b) 8 : 4

   II (a) 8 × 0.4 (b) 8 : 0.4

   III (a) 0.8 × 0.4 (b) 0.8 : 0.4

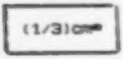
   IV (a) 0.8 × 4 (b) 0.8 : 4

   (Parts I-III of this question were taken from Brown, 1981, who examined 12-15 year old students. We added part IV to them in order to complete the structure.)
8. How will you explain to a student which of the symbols: < , > , = should be written between the two numerical expressions without carrying out the computations? Please, explain your answer!

   I $\frac{4}{3} \times \frac{2}{5}$ . . . $\frac{2}{5}$ II $\frac{3}{4} : \frac{3}{2}$ . . . $\frac{3}{4} \times \frac{2}{3}$

   III 8 × 7 . . . 8 : 7
9. A rectangle whose area is 1/3 cm² is given. The length of one of its sides is 3/5 cm. What is the length of the adjacent side?

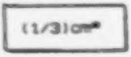
Figure 1



(This question was given only to half of the sample. It was taken from Hart, 1981. The other half of the sample got Question 10 for the sake of comparison.)

10. A rectangle whose area is 1/3 cm^2 is given. The length of one of its sides is 5 cm. What is the length of the other side?

Figure 2



The reader can see that Questions 1-6 are related to the models of division and Questions 7-10 are related to the views about multiplication and division possibly or partly implied by these models.

Sample

The above questionnaire was distributed to 237 teachers and 72 pre-service teachers. 54 teachers out of the 237 had the official title of Mathematics coordinators in their schools. These are teachers who have more interest in Mathematics than the average teacher and also underwent some in-service mathematical training. In the result section they will be referred to as Math. coordinators while the other teachers will be referred to as teachers.

Results

The answers of the respondents were analysed and classified to some main categories. Questions 1-5 were supposed to expose the models of the respondents for the division operation. More precisely, there was an attempt in the questions to direct the respondents toward the formal approach. Formal approach can be understood in two ways: 1. Not concrete: namely, no reference to quotative or partative division. 2. In addition to 1, conceiving the division as the inverse operation of multiplication. Hence in Questions 1-5, the formal approach and the concrete models (partitive and quotative divisions) play a central role. There are special categories in some of the questions, resulting from the special situations in these questions. The information is given in Table 1.

Table 1
Models for Division
Distribution of answers (in percentages) to Ques. 1-5 in the three groups.
T-teachers(N=152),P-preservice teachers(N=72),M-Math.coordinators(N=34)

Category	Formal			Partitive only			Quatative only			Partitive or Quatative			A Category Specific to the Question (see below)			No Answer and Other		
Question	T	P	M	T	P	M	T	P	M	T	P	M	T	P	M	T	P	M
1	24	7	29	26	11	29	11	27	29	9	12	[illegible]	15	22	8	11	12	17
2	21	22	27	14	12	19	12	8	19	-	-		6	9	0	45	61	35
3				26	19	16	33	42	37	34	19	47						
4	14	6	27							59	59	8				40	35	65
5	52	21	66							22	22	[illegible]				26	48	11

The specific categories for Questions 1,2 were:
Question 1: The models for division are restricted to small numbers. For big numbers, division is mechanical.
Question 2: Explanations for division by 3 instead of division by 1/3 (in other words, division by 1/3 is understood as division by 3).

We would like to illustrate the categories by some quotations. Because of lack of space we do not bring here their analysis. This will be given elsewhere.

Question 1: (*) *Division can be used as the inverse operation of multiplication. By what should we multiply 84 in order to get 381* (The formal approach).

(*) *We want to know how many times there are 84 in 381* (Quatative division).

(*) *381 is consisted of 381 partial numbers which should be divided to 84 sets* (Partitive division).

(*) *You do not illustrate the meaning of division by means of large numbers. The goal is to explain what division is. One should stay at the range of small numbers* (The specific category for this question).

(*) *You should explain what is 8 divided by 4. 381:84 is carried out automatically* (The specific category for this operation).

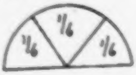
Question 2: (*) *(1/2):(1/3) is a formal operation only. This is because division by a fraction appears as multiplication (1/2)x(3/1).* (The formal approach.)

(*) *One half of a cake is given to one third of a person. Since every person is a whole, he or she gets one cake and a half.* (Partitive division.)

(*) *(1/2):(1/3) is dividing half a cake to one third of a class* (Partitive division.)

(*) *How many times does 1/3 go into 1/2.* (Quatative division.)

(*) *If I have a half of a certain quantity, like half a cake, I divide by 3 and each part is 1/6. (1/2) : (1/3) = 1/6.*



(The specific category for this question)

(*) *Half a cake was left in the refrigerator. I gave 1/3 of it to each of my children.*(The specific category for this question.)

Question 3: The correct answer to this question is, of course, that 18:3 is neither partitive nor quotative. 18:3 has the potential to be either partitive or quotative, it depends on the situation where it is used. Such answers were classified as "partitive or quotative" in Table 1. In the other answers it was claimed, that 18:3 was either partitive or quotative, but cannot be both. Namely, 66% of the teachers, 81% of the preservice teachers and 53% of the Math. coordinators did not demonstrate in this question the understanding that division is an abstract operation and partitive and quotative divisions are two of its concrete models.

Question 4: (*) *Division by zero is meaningless.* (The formal approach.)

(*) *Division by 0 is not permitted. The answer is not reasonable* (Formal.)

(*) *This is a meaningless expression Every division exercise can be checked by a multiplication exercise. For instance: 6 : 2 = 3, 6 = 2 x 3. But 5 : 0 = ?, 5 = 0 x ?. Every number multiplied by 0 is 0 and not 5.* (Formal.)

(*) *5 : 0 = 5. To divide five cakes to 0 children, I'll be left with five.*(Partitive)

(*) *5 : 0 = 0. 0 represents here nothing. Therefore, division by nothing of any number is 0.* (Partitive or quotative.)

Question 5: (*) *Essentially the student is right. However, in the case of 0/0 it is meaningless because it can be any number, even 8, 0/0 = 8, 0 x 8 = 0.*(Formal).

(*) *0/0 = 1 because 0 x 1 = 0.* (Formal.)

(*) *0 is not like other numbers.* (Formal.)

(*) *0 of something divided by 0 children gives 0.*(Partitive.)

(*) *0 has no numerical value. therefore it is impossible that division by 0 will give a numerical value. The answer must be 0.* (Partitive or quotative.)

(*) *(0/0) = 1 because 0 is less than 1 and when you divide the result should be less than the dividend.* (Partitive or quotative.)

(*) *It is wrong. You should always ask the division question: how many times the divisor is in the dividend. When you consider 0, the answer to the question "how many times ther is 0 in 0?" is 0.* (Quotative.)

Question 6 was designed to examine the relation between division and other arithmetical operations (multiplication, repeated addition and repeated subtraction). Distractor (b) was an irrelevant distractor whereas in all the other distractors there was a real offer. The results are given in Table 2.

Table 2
Distribution of answers to Question 6.
The relation between division and other arithmetical operations
(The numbers indicate percentages)

Category Group	Multiplication (distractors (a) and (e))	Repeated addition (distractor (c))	Repeated subtraction (distractor (d))	Distractor (d)	No answer
Teachers (N = 183)	30	16	13	3	38
Preservice Teachers (N = 72)	42	16	3	6	33
Math.Coor-dinators (N = 54)	50	16	5	3	26

We would like to note that the only case where respondents chose more than one distractor was the combination of (a) and (e). No other combination has been found. Thus, in the context of this question, the percentages of those who are aware of the special relation between division and multiplication in the three groups are 30, 42, and 50, respectively.

The analysis of the answers to Questions 7-10 is organized in a similar manner to those of Question 1-5. Because of lack of space we will not illustrate the answer categories by quotations.

Table 3

Distribution of answers (in percentages) to Ques. 7-10 in the three groups.
T-teachers(N=150),P-preservice teachers(N=72),M-Math.coordinators(N=54)

Category	A correct answer with a correct or even partially correct explanation			A category specific to the question (see below)			Multiplication makes bigger and division makes smaller			A wrong answer based on other principles			No answer		
Question	T	P	M	T	P	M	T	P	M	T	P	M	T	P	M
7	26	9	46	27	29	27	10	24	0	17	23	19	20	15	8
8	34	27	46	22	27	27	16	23	8				28	24	19
9	53	47	77				12	15	12	6	8	3	29	29	8
10	32	51	66				40	26	13	9	3	0	19	20	14

Note that Question 9 was administered to half of the sample and Question 10 to the other half.

The specific categories for Questions 7,8 were:

Question 7: The answer was given after a computation was carried out. There was no attempt to establish the answer on general arguments like: multiplication by a (possibly) fraction smaller than 1 makes smaller.

Question 8: Multiplication by a fraction always makes smaller and division by a fraction always makes bigger (An overgeneralization of the case of proper fractions)

Discussion

As we explained in the introduction, our goal in this study was to verify and examine directly some claims about models for division and some views about division and multiplication, claims which were made by indirect methods using psychological interpretation of certain data (Fischbein et al., 1985). We found that these claims were basically correct but the situation is much more complex than it is described in Fischbein et al., 1985 and Tirosh et al., 1986, 1987. In Fischbein, 1985, the elementary teachers are treated as if they have the required mathematical knowledge. ("Teachers of arithmetic face a fundamental didactical dilemma.... This is one instance of a general dilemma facing mathematics teachers". p.15, there, last paragraph.) This study and also previous ones (as Tirosh et al., 1986, 1987) clearly show that Fischbein's implicit assumptions have no ground. The elementary teachers, as a group, lack basic mathematical understanding of arithmetic.

References

- Bell, A., Swan, M., & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. **Educational studies in Mathematics**, 12, 399-420.

- Brown, M. (1981). Number Operations. in Hart, K.M. (Ed.). **Children's Understanding of Mathematics**: 11-16. London: John Murray, 23-48.

- Fischbein, E., Deri, M., Nello, M.S., & Marino, M.S. (1985). The role of implicit models in solving verbal problems in multiplication and division. **Journal for Research in Mathematics** Education, 16, 3-17.

- Hart, K.M. (Ed.) (1981). **Children's Understanding of Mathematics: 11-16.** London: John Murray.

- Hempel, C.G. (1966). **Philosophy of Natural Science.** Prentice Hall.

- Hershkowitz, R., & Vinner, S. (1984). Children's concepts in elementary geometry: a reflection of teachers' concepts? **Proceedings of the 8-th International Conference for the Psychology of Mathematics Education.** Sidney, 63-69.

- Hershkowitz, R., Bruckheimer, M., & Vinner, S. (1987). Activities with teachers based on cognitive research, **NCTM 1987 Yearbook**, 222-235.

- Hiebert, D., & Hiebert, J. (1983). Junior high School students' understanding of fractions. **School Science and Mathematics**, 83, 96-106.

- Tirosh, D., Graeber, A.O., & Glover, R. M. (1986). Preservice teachers' choice of operation for multiplication and division word problems. **Proceedings of the 10-th International Conference for the Psychology of Mathematics Education.** London, 57-62.

- Tirosh, C., Tirosh, D., Graeber, A.O., & Wilson, J.M. (1987). Interventions to correct preservice teachers' misconceptions about the operation of division. **Proceedings of the 11-th International Conference for the Psychology of Mathematics Education.** Montreal, 3, 369-375.

THE INFLUENCE OF SOCIALIZATION AND EMOTIONAL FACTORS ON MATHEMATICS ACHIEVEMENT AND PARTICIPATION

Delene Visser, University of South Africa

This study concerns the explanation of sex differences that typically occur from adolescence onwards and favour males in achievement and participation in mathematics. In the absence of conclusive biological evidence, social, emotional, and attitudinal factors were investigated in this regard. The subjects were 1 605 Afrikaans-speaking seventh and ninth grade students and 2 506 of their parents. Cognitive measures included mathematics achievement and several aptitude tests. Also measured were attitudinal variables such as confidence and enjoyment of mathematics, perception of the attitudes of significant others towards self, personal and general usefulness of mathematics, and the stereotyping of mathematics. For ninth grade students, but not for seventh grade students, significant differences favouring males were found in spatial abilities and several attitudinal variables. The intention to continue participation in mathematics was accurately predicted by attitudinal variables in the case of ninth grade females, but not males.

Adequate preparation in mathematics has aptly been called the 'critical filter' in the job market. As a result of technological advances and the information explosion, a certain degree of mathematical sophistication has become a prerequisite for most prestigious occupations. Students who elect to discontinue their mathematics studies while they are still at school thereby effectively eliminate themselves from the majority of better paid occupations. Furthermore, in a developing country such as South Africa where every effort should be made to alleviate the shortage of scientific, research, and technical personnel, an obvious starting point is to ensure that as many students as possible complete the mathematics courses offered at school.

In South Africa mathematics is compulsory until the ninth grade, whereafter students may opt either to discontinue their mathematics studies, or to continue until the twelfth grade. The far-reaching decision to discontinue school mathematics is therefore made by 14 to 15 year old adolescents, which makes it especially important to establish which factors influence their decision during this period.

It has been reported frequently that no or few sex differences in mathematics achievement or ability are evident until the age of about 13 years, whereafter the performance of females begins to decline in relation to that of males, especially in areas such as problem solving (Armstrong, 1985; ETS, 1979; Husén, 1967; Maccoby & Jacklin, 1974; Preece, 1979; Wise, 1985; Wood, 1976). At the upper end of the achievement scale it seems that sex differences favouring males are even more pronounced. Benbow and Stanley (1980, 1983) reported sex differences among mathematically gifted students from about the seventh grade, while the ETS (1979) report also confirmed superior performance by males among the top scorers on the Mathematics SAT. Males are also far more likely to enrol in high school mathematics courses than are females (Fennema & Sherman, 1977; Sells, 1978; Wise, Steel, & MacDonald, 1979).

In South Africa similar tendencies are found. During 1980 72% of the twelfth grade males as against 48% of the females in the Transvaal (white population only) studied mathematics. It should be remembered that not all of these students passed mathematics or attained levels of achievement which would have allowed them access to mathematics-related university or technikon courses. The corresponding figures for 1984 were 84% for males and 62% for females. It is gratifying to note that the position has improved for both sexes, but the fact remains that notable sex differences in school mathematics participation still exist in South Africa.

With regard to achievement, no noteworthy sex differences in twelfth grade final mathematics examination results were found. Among the top scorers, however, males predominate. During 1982, 4,6% of the males as against 3,5% of the females scored over 80%, whereas the corresponding figures for 1984 were 3,2% for males and 2,6% for females. A nation-wide mathematics olympiad is arranged annually for mathematically gifted students. In the period 1966 to 1985 only 12 females gained silver medals as against the 183 silver medals awarded to males. No gold medal has yet been awarded to a female, and in 1986 only five females as against 98 males progressed to the final round of the olympiad.

The differentiation in mathematical functioning between males and females which is manifested from early adolescence onwards, needs to be explained in terms of developmental changes which occur during this life period. In the absence of conclusive biological evidence to explain the said differences, it was decided to investigate the role of affective, motivational, and socialization factors in this regard.

Although sex-role socialization starts at birth, it is from early adolescence onwards that sex-appropriate behaviour is increasingly expected from males and females (Mussen, Conger, Kagan & Huston, 1984). Mathematics has traditionally been regarded as a male domain, because so few women have distinguished themselves in this field. Even in recent years mathematics is stereotyped as a male domain, particularly by adolescents (Brush, 1980; Ormerod, 1981).

Important socializers such as parents, peers, and teachers put pressure on adolescents to conform to sex role standards (Mussen et al., 1984). It is therefore to be expected that males would be encouraged in mathematics, whereas females would be discouraged in a variety of subtle ways. Females may consequently develop anxiety about mathematics achievement and feel less motivated than males to participate in the subject.

METHOD

Seventh grade students were selected to represent the early adolescent group in this study, whereas ninth grade students were selected to represent the adolescent group. The students were randomly selected from Transvaal Afrikaans schools after stratification by sex and rural-urban location. Thirty-six high schools and 36 primary schools were included in the study. The parents of each student were also invited to participate in the study, and almost 80% of them agreed to participate. The sample consisted of 824 seventh grade students and 781 ninth grade students. Altogether 1 186 fathers and 1 320 mothers participated in the study. The mean age on the first day of testing for seventh grade students was 12,4 years and for ninth grade students 14,4 years.

The measuring instruments included standardized aptitude tests, mathematics achievement tests developed specially for this study, translated versions of American questionnaires, and questionnaires developed with this study in mind.

Six subtests of the Junior Aptitude Tests (JAT), standardized by the Human Sciences Research Council (HSRC), were used for measuring verbal and reasoning ability, numerical ability, and spatial visualization.

Two mathematics achievement tests, one for seventh and one for ninth grade, were developed especially for this study by the HSRC. The tests were based on the students' current mathematics curricula.

Eleven Likert-type attitude scales were developed and/or translated to measure students' and parents' attitudes to mathematics. The first four scales mentioned below are similar to the Fennema-Sherman Mathematics Attitudes Scales (Fennema & Sherman, 1976) with the same titles and were developed by Visser (1983). Items adapted from Aiken's E- and V-scales (1974), items from the Fennema-Sherman Scales, and several original items, were included in the final scales.

The Confidence Scale was developed to measure a subject's confidence versus his/her discomfort, anxiety, and uncertainty when dealing with mathematics.

The Motivation Scale measures a subject's interest in and willingness to become more deeply involved in mathematics.

Other scales were the Male Domain, General Usefulness, Personal Usefulness and Attitude toward Success Scales.

The Perception of Father's (Mother's, Teacher's, Male Peer Group's) Attitude Scales were included to measure the perceived interest and encouragement from significant others.

The Importance for 'X' Scale measures the importance attached by parents to their child's mathematics studies and the degree to which they encourage the child.

The attitude scales were scored such that a high score indicates a positive attitude toward mathematics. On the Male Domain Scale a low score is indicative of the stereotyping of mathematics as a male domain.

RESULTS

Interesting results included the following:

As early as grade 7 more males than females intend to persist with mathematics until the twelfth grade. In the ninth grade 94 percent of the males as against 65 percent of the females indicated that they wished to complete their mathematics studies. In accordance with the findings of overseas studies, a clear picture failed to emerge for sex differences in mathematics achievement over the entire range of the achievement scale. It was shown that in the USA males predominate at the top end of the scale, but also that males do not usually obtain higher school marks in mathematics. In the present study t tests using sex as independent variable were performed on each of the student variables. No sex differences were found on the mathematics achievement tests in either of the grades. Furthermore, no sex difference was found in either of the two standards on the computation test, JAT Number.

It has been hypothesized that sex differences in mathematics achievement may be explained by sex differences in spatial orientation and visualization which are also typically found from adolescence onwards (Connor & Serbin, 1980; Fennema & Sherman, 1977; Maccoby & Jacklin, 1974). No sex differences were found for seventh grade students on the JAT Spatial 2-D and Spatial 3-D tests, whereas significant differences on these tests favouring males were found for ninth grade students. If sex differences had been found on the achievement tests, particularly with regard to certain branches of mathematics, the obtained sex differences in spatial visualization for ninth graders might have provided an explanation.

As far as the attitudinal variables were concerned, it was found that seventh and ninth grade males were more inclined than their female counterparts to regard mathematics as personally useful and had a more positive perception of the male peer group's attitude toward themselves as learners of mathematics. Males were also more inclined to stereotype mathematics as a male domain. However, on the Confidence, Motivation, Father and

Mother Scales significant sex differences favouring males were found only in the older age group.

Although parents agreed on the general usefulness of mathematics, fathers had higher scores than mothers on the other attitudinal variables. Both mothers and fathers regarded mathematics as more important for their sons than for their daughters.

These findings lend support to the hypothesis that developmental changes caused by the environment during adolescence may be partially responsible for sex differences in mathematics participation.

It was decided to use a purely predictive model rather than a 'causal' model for determining the relationships between the various cognitive and attitudinal variables and the dependent variables, mathematics achievement and mathematics participation. Step-wise multiple regression analyses were performed for each grade and sex for mathematics achievement as dependent variable and for intended participation as dependent variable.

It was found that cognitive variables are the best predictors of mathematics achievement during the seventh and ninth grades, but that several attitudinal variables and some parent variables correlate highly with the achievement of ninth grade females.

Furthermore, attitudinal variables predominated over cognitive variables as predictors of intended mathematics participation. It was found that the pattern of high correlations varied according to the sex and grade of students. In the case of males, especially ninth grade males, very few variables correlated highly with intended participation, whereas cognitive variables seemed to be almost irrelevant. Only Personal Usefulness had a substantial correlation with intended participation for ninth grade males. The low squared multiple correlations (0,33 and 0,26) reflected the above observations. The decision of adolescent males to continue their participation in mathematics is therefore to a large extent taken independently of the study variables.

A different picture emerged for females. Several student attitude variables correlated highly with intended participation for both grades, whereas some cognitive variables and the Importance for 'X'

(Father and Mother) Scales also correlated substantially with the dependent variable for ninth grade females. Five attitude variables and one cognitive variable accounted for as much as 65% of the variance of ninth grade females' intended mathematics participation, whereas the squared multiple correlation for seventh grade females was only 0,41. Encouragement of parents seemed to be a major influence on the mathematics behaviour of adolescent females. For males as well as females, perceived personal usefulness of mathematics was the strongest predictor.

DISCUSSION

The purpose of this study was to identify and explain the factors affecting mathematics participation and achievement during adolescence and, in particular, to find explanations for sex differences which typically occur from adolescence onwards in mathematics behaviour.

Developmental changes do seem to occur in the period between early adolescence and adolescence which negatively affect the affective and attitudinal position of females with regard to mathematics, as well as their perception of the expectations and encouragement of significant others.

The findings of this study support the view that early adolescence is a critical period during which achievement patterns in mathematics are established, with almost inevitable implications for future vocational options.

REFERENCES

Aiken, L.R., Jnr. (1974). Two scales of attitude toward mathematics. *Journal for Research in Mathematics Education*, *5*, 67-71.

Armstrong, J.M. (1985). A national assessment of participation and achievement of women in mathematics. In S.F. Chipman, L.R. Brush, & D.M. Wilson (Eds.), *Women and mathematics: Balancing the equation* (pp. 59-94). Hillsdale, NJ: Erlbaum.

Benbcw, C.P., & Stanley, J.C. (1980). Sex differences in mathematical reasoning ability: Fact or artifact? *Science*, *210*, 1262-1264.

Benbow, C.P., & Stanley, J.C. (1983). Sex differences in mathematical reasoning ability: More facts. *Science*, *222*, 1029-1031.

Brush, L.R. (1980). *Encouraging girls in mathematics: The problem and the solution*. Cambridge, Mass.: Abt Associates.

Connor, J.M., & Serbin, L.A. (1980). *Mathematics, visual spatial ability and sex roles*. Washington, DC: National Institute of Education.

Educational Testing Service. (1979). *National college-bound seniors*. College Entrance Examination Board.

Fennema, E. (1977). Influences of selected cognitive, affective, and educational variables on sex-related differences in mathematics learning and studying. In *Women and mathematics: Research perspectives for change* (NIE Papers in Education and Work No. 8, U.S. Department of Health, Education, and Welfare). Washington, DC: U.S. Government Printing office.

Fennema, E., & Sherman, J. (1976). Fennema-Sherman Mathematics Attitudes Scales. *JSAS Catalog of Selected Documents in Psychology*, *6*(1), 31. (Ms. No. 1225)

Fennema, E., & Sherman, J.A. (1977). Sex-related differences in mathematics achievement, spatial visualization and affective factors. *American Educational Research Journal*, *14*(1), 51-71.

Husén, T. (Ed.). (1967). *International study of achievement in mathematics* (Vols. 1 & 2). New York: Wiley.

Maccoby, E.E., & Jacklin, C.N. (1974). *The psychology of sex differences*. Stanford, CA: Stanford University Press.

Mussen, P.H., Conger, J.J., Kagan, J., & Huston, A.C. (1984). *Child development and personality*. New York: Harper & Row.

Ormerod, M. (1981). Factors differentially affecting the science subject preferences, choices and attitudes of girls and boys. In A. Kelly (Ed.), *The missing half: Girls and science education*. Manchester: Manchester University Press.

Preece, M. (1979). Mathematics: The unpredictability of girls. *Mathematics Teaching*, *87*, 27-29.

Sells, L.W. (1978). Mathematics: A critical filter. *Science Teacher*, *45*(2), 28-29.

Visser, D. (1983). The development and comparison of semantic differential and Likert-type scales to measure attitudes to mathematics. *Reports from the Psychology Department*, University of South Africa, No. 9.

Wise, L.L. (1985). Project TALENT: Mathematics course participation in the 1960s and its career consequences. In S.F. Chipman, L.R. Brush, & D.M. Wilson (Eds), *Women and mathematics: Balancing the equation* (pp.25-58). Hillsdale, NJ: Erlbaum.

Wise, L.L., Steel, L., & MacDonald, C. (1979). *Origins and career consequences of sex differences in high school mathematics achievement*. Palo Alto, CA: American Institutes for Research.

Wood, R. (1976). Sex differences in mathematics attainment at GCE Ordinary Level. *Educational Studies*, *2*(2), 141-160.

METACOGNITION AND ELEMENTARY SCHOOL MATHEMATICS

Miriam A. Wolters, Department for Developmental Psychology. State
University Utrecht. The Netherlands.

Abstract

Recent research on cognitive development, memory, reading and mathematics indicates that much attention is given to metacognition. This paper is intended as an introduction to the operationalisation of metacognition and the role elementary school mathematics plays in metacognitive development.

The longitudinal study assessed the effects of two approaches in school mathematics on the development of metacognitive skill. In each condition 15 students were followed from the first through the fourth grade. During these years they were tested four times in order to assess the developmental level of metacognitive skill. The data are analysed by trend- and t-test analysis and the results are discussed.

Introduction

The basic purpose of the study is to develop instruments for measuring metacognition and to determine the effects of elementary school mathematics on metacognition. In recent years metacognitive processes during mathematical problem solving have become an importane topic of discussion in mathematics education (e.g. Garofalo, lambdin Kroll & Lester 1987; DeGuire 1987; Hart 1987). However, none of these studies look at metacognitive functioning in students aged 6-10. Therefore, in this paper we refer to studies of metacognition as a developmental phenomenon. Two categories of metacognitive activities are mentioned: (1) those concerning conscious reflection on one's own cognitive activities and abilities, and (2) those concerning self-regulatory mechanisms going on during an attempt to learn to solve problems (cf. Wertsch 1985).

In this paper we are concerned with this second category of activities. This category involves content-free strategies or procedural knowledge such as self-interrogation skills, selfchecking, and so forth. In other words it is an activity by means of which the learner manages his (or her) own thinking behavior.

A central problem in the research on metacognition is the adequacy of assessment techniques designed to measure metacognition. Meichenbaum, Burland, Gruson & Cameron (1985) consider several different techniques that can and have been employed to study metacognitive activities in children. They point out that one of the pitfalls of the interview and think-aloud techniques is that the interpretation of the data yielded by such techniques is problematic. The most serious problem here arises when a subject has trouble verbalising his answers or thinking pattern. The absence of an adequate response does not necessarily mean that subjects are not involved in metacognitive activities. For example, Grumon demonstrated on the basis of observations that there are subjects who show consistent strategies, but who fail to verbalize such strategies. The same phenomenon was also observed in Burland's and Cameron's data. Thus, the use of interview and think-aloud techniques raises an important theoretical issue: do we indeed limit the definition of metacognition to the subject's abilities to verbalize their thinking process?

A somewhat different approach without the above mentioned pitfalls is to assess metacognitive involvement directly on the basis of performance without the subject reporting his thinking process either during performance or afterwards. Grumon (1985) has shown that it is possible to infer the use of metacognitive strategies on the basis of repeated patterns evident while carrying out the task. Examples of how one can formally conduct metacognitive assessment without using self-reports come from the work of Sternberg (1983), Butterfield, Wambold & Belmont (1973) and the Sovietpsychological work of Isaev (1984) and Zak (1986).

In our study we further develop the line of investigation introduced by the Soviets, i.e. conducting metacognitive assessment directly on performance, thus making less use of verbal questioning and focusing more on behavioral observations. The Soviets see the issue of reflective thinking or metacognition as a continuum beginning with manipulative strategies and eventually progressing through empirical towards more theoretical strategies. A manipulative strategy consists of moves that are not guided by the goal. Such a move does not logically follow subject's preceding move and neither is it the basis for the next move; the moves are not connected. Most often a large number of superfluous moves is needed to reach the end result. Subjects using an empirical strategy approach the task through moves or actions that change the situation step by step. The subject expects a specific result from a move and takes that into account when making the next move. Subjects using a theoretical strategy think over their solution beforehand. These subjects use the first and second item to search for most efficient way of solving the task by testing in their mind sometimes three or four non-optimal alternative strategies.

In this study the tasks measuring metacognitive skill are designed in a such a manner that the observer is allowed to draw inferences about the level of metacognitive functioning. The non-mathematical task is constructed to elicit different solving strategies. An integral part of the task are the specific procedures for scoring the different strategies a subject uses when solving a given task. The tasks and scoring procedures are designed in such a way that subjects who change strategies can be identified as well. Subjects require no special knowledge and are not familiar with the tasks. The subjects apparently like to do the tasks. They can not fail them because they are constructed in such a way that nobody can do them wrong; the only thing that matters is the way in which the subject handles the task.

METHOD

Subjects

Students from four middle class schools participated in the longitudinal study. The schools were choosen because of their willingness to participate. Two schools followed a traditional mathematics curriculum and formed the so called control condition. The other two schools followed an experimental structuralistic mathematics curriculum and formed the experimental condition. 15 students were selected from the control condition and 15 comparable students were selected from the experimental condition on the basis of a pretest score, administered when they entered first grade.

Procedure

When entering first grade students were pretested to assess a general cognitive developmental level. The pre-test was administered by the class teachers, but always with an experimenter present. The pretest score was used to arrange two matched groups of 15 students. The mathematic activities then took place during the regularly scheduled mathematics lessons from the first through the fourth grade. The traditional program was given in all four grades for the control group and in grade 3 and 4 of the experimental group. Only in grade 1 and 2 an experimental structuralistic approach was used. Four times in total the selected groups of students were tested on metacognition. Twice in the second grade, in December the Token task and in June the Hole task. In the third grade in June the Strip task and in June of the fourth grade the Token task.

Instruments

1. The pretest. The pretest was designed to assess children's abilities in combining classification and seriation.

2. Assessment of metacognitive skill. We used three tasks: the Token task, the Strip task and the Mole task. Each task consists of 8 items. After four items a moment of reflection for the subject is induced. This happens indirectly by way of a special instruction. In the Strip task a reflection moment is induced by indicating to the subject the need to think before solving the task which is timed. After four items of each task the subject is given the opportunity to think about the efficiency of the strategy used and possibly change the strategy to a more efficient one. In the Token task the activity is to make a pattern of tokens similar to a given pattern in a minimal number of moves. In the Mole task the activity is to find the shortest route of a mole to his feeding place in a structured garden.

Instruction and scoring of these tasks will be illustrated by a detailed description of one of the tasks: the Strip task.

STRIP TASK

The strip task was originally developed by the Soviet psychologist Zak (in Wolters 1987) and was designed to measure reflection as a metacognitive skill.

The material used is a board with an area of 30 x 60 cm on which two parallel lines, with a distance of 15 cm.

Strips are used in the following numbers and measures:

stripindex	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
length	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48
number	10	10	10	5	5	5	3	2	1	1	1	1	1	1	1	1

The length is given in cm. All strips are 3 cm wide

INSTRUCTION

The instruction consists of two phases. In the first phase the subject is shown a model strip and asked to make up a strip of the same length as the model. The subject is given a number of strips of varying length and then told to use a specified number of strips for constructing a length equal to the model. It is emphasized that he has to think carefully before setting out to solve the task.

Before starting the task-items two introductory items are presented: first a model strip with a length of 9 units is presented and the subject is instructed to build a matching strip using two parts. The item is coded as 9(2); the 9 indicating the length of the model and the (2) indicating the number of parts to be used in matching the model. Task items for the first phase are: 10(4), 14(5), 13(6), 12(7)

After the subjects have done four items they are given instructions for the second phase. These are designed so as to encourage them to think about the task before they actually begin selecting the strips to match the model. They are told "from now on we will see how much time you need to do a strip". The subjects are told that they can take as much time as they want to think about the problem and that they will be timed only when they begin selecting and placing the strips. For this phase four additional items are presented to each student. This second phase is used to determine if students change the strategy they used in the first phase as a result of instructions given prior to the second phase items. Performance time is taken for items: 16(9), 15(8), 11(7) and 13(6). One item

13(6) is used twice, once before time instruction and once after time instruction. This item is meant as an extra check to see if subjects change their strategy.

SCORING

ITEM SCORING

Manipulative category (includes scores 1, 2 and 3)

This category includes behaviors that are hapnazard and without any planning. The subject is unaware of the end result until after it has been accomplished. It is only at that time that the subject recognizes that the task is completed. The subject behaves according to the rules while attempting to match the model in length but looses track of the requested number of strips. The subjects in this category are characterized by placing and replacing the strips ("removing behavior") eventually coming to use the correct number of strips by less removing behavior. Score 1 means that they end up with an incorrect number of strips. The difference between score 2 and 3 is the number of strips removed and replaced.

Empirical category (score 4 and 5)

This category implies that a subject has in mind a strategy characterized as inductive which means that the subject recognizes the goal of the task. The subject has no need to remove strips once they are placed, but rather adjusts the size of the strips as the task is being solved. The subject behaves in a step by step fashion, placing one or two strips, making a decision, placing another strip and adjusting the next and continuing in this fashion until all the strips are correctly placed. The difference between score 4 and 5 is that more steps are used for score 4 than 5.

Theoretical category (score 6 and 7)

The behavior in this category is the most efficient since the subject proceeds in a deductive manner. The subject does all the planning prior to the moment he actually puts the strips in place. In this fashion the subject takes a stack of strips one less than the necessary number, places these in correspondence to the model and then determines the size of the last strip completing the comparison. With score 6 an estimation error is made with the completing strip.

CODING OF METACOGNITIVE LEVELS

For each of the three tasks a score-level was calculated by taking the mean score over the eight items. Apart from a score-level a so called code-level was calculated, indicating the effect of the moment of reflection induced halfway each task. The procedure to obtain the code-level for each task is as follows: for each of the three tasks for the first four items and the second four items scores were placed in one of the three categories: manipulative - empirical - theoretical. It was then possible to obtain a coded score for each subject on each of the three tasks based on whether or not the strategy changed from the first to the second phase of each task. These coded scores were placed in a numerical hierarchy from 1 to 7 with a code-level of 1 demonstrating the strategies using the least metacognition and a code-level of 7 demonstrating the most. For example, a subject with a code-level of 1 on the strip task would have used a manipulative strategy for the first four items and continued with a manipulative strategy for the four items after reflecting was requested. A subject coded 6 uses an empirical strategy for the first four items and changes to a theoretical strategy for the second four items. Fig.1 illustrates the seven code-levels that were used. A student with a mean score smaller than 3.50, falling between 3.50-5.00, or greater than 5.00 was classified as manipulative, empirical or theoretical respectively. The criteria for change from the first four items to the second four items is that the difference between the mean score achieved on the second four items had to be equal or greater than 0.75 than the mean score achieved on the first four items. In addition the mean score for the second four items had to fall in a category above the mean score of the first four items.

	mean score item 1-4		mean score item 4-8
code 1	manipulative	remains	manipulative
code 2	empirical	changes	manipulative
code 2	theoretical	changes	manipulative
code 3	manipulative	changes	empirical
code 4	empirical	remains	empirical
code 5	theoretical	changes	empirical
code 6	empirical	changes	theoretical
code 7	theoretical	remains	theoretical

fig.1 Calculation of the code-levels

3. Elementary school mathematics curricula

In the control condition the teachers used a traditional arithmetic program. In the experimental condition this traditional arithmetic program was used from the third grade onwards. In the first and second grade an experimental mathematical program was used. This experimental program has a structuralistic nature and is very much inspired by the Soviet psychologist Davydov. The program consists of three main structures: numeration system, operations and relations. In the first grade the three structures are taught separately and in the second grade the students learn to integrate them when learning to add and subtract two-digit numbers. The numeration system of the program is described in Wolters (1986a), the operations part in Wolters (1986b). The part on relations follows a line of thinking introduced by Davydov (1962).

RESULTS

To measure metacognitive skill validated instruments are needed. The procedures and tasks were validated in another study. In that study we computed with a group of elementary schoolchildren correlations between the three tasks and the pretest. The correlations are: pretest with Token task .31 (p=.09); pretest with Strip task .52 (p=.009); pretest with Mole task .68 (p=.001). The correlation between Token and Strip task is .50 (p=.01); Token and Mole task .73 (p=.001); Strip and Mole task .69

(p=.001). This means that the metacognitive measures are highly related to each other and for two of the three metacognitive tasks also highly related to the pretest.

The results of a trend analysis of the changes in metacognitive code per condition (experimental versus control) is depicted in fig.2. Fig.2 shows that the metacognitive code in the control group increases with age and years of mathematics instruction. For the experimental group a different picture emerges. At the first measurement in the second grade this group tends to perform better on metacognitive tasks. Here the experimental group outperformes the matched control group. But at the second measurement in the second grade the difference decreases and dissappears completely at the end of the third grade. These results indicate that the students in the experimental group have developed their metacognitive abilities through working with a structuralistic mathematics program in the first grade. The effects do not last long. This can be explained first of all by the fact that after two years the experimental group goes back to a traditional program. Secondly although the method of teaching still differs during the second grade both groups learn to add and subtract two-digit numbers. As the teachers have to teach material that they are familiar with they easily fall back on well known teaching methods. So even during the latter part of the second grade the metacognitive lead of the experimental group diminishes rapidly.

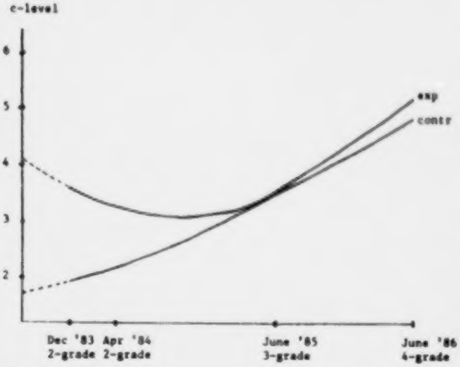


fig.2

	1-measur		2-measur		3-measur		4-measur	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
exp	4.22*	1.56	2.44	1.01	4.00	1.50	5.33	2.00
contr	2.11	1.17	2.13	1.15	3.66	.70	5.00	2.00

Table 1 Means and standard deviations for metacognitive code-level for experimental and control group

A t-test analysis on the data of table 1 shows a significant difference between experimental and control group at the first measurement in the second grade (p=.003)

REFERENCES

Butterfield E., Wambold C., & Belmont J. (1973). On the theory and practice of improving short-term memory. American Journal of Mental Deficiency, 77, 654-669.

Davydov V.V. (1962), An experiment in introducing elements of algebra in elementary school. Soviet Education 5,27-37

DeGuire L.J. (1987), Awareness of metacognitive processes during mathematical problem solving. In: Proceedings of the XI conference on Psychology of mathematics education. Montreal july 19-25, Vol.II

Garofalo J., D. Lambdin Kroll & F. Lester, Metacognition and mathematical problem solving: Prelimenary research findings. In: Proceedings of the XI conference on Psychology of mathematics education. Montreal july 19-25, Vol.II

Hart L. (1987), Interactive domains, metacognition and problem solving. In: Proceedings of the XI conference on Psychology of mathematics education. Montreal july 19-25, Vol.II

Isaev K.J. (1985), Psychological description of young childrens planning methods. Soviet Psychology 1,35-50

Meicnenbaum D., Burland S., Gruson L., & Cameron R. (1985). Metacognitive assessment. In S.R. Yussen (ed.), The growth of reflection in children. A.P., London.

Sternberg R.J.(1983) Components of human intelligence. Cognition, 15, 1-48.

Wertsch J.V. (1985), Adult Child interaction as a source of selfregulation in children. In: Yussen, S.R. (ed.) The growth of reflection in children. Academic Press

Wolters M. (1986a), Is 89 larger than 91? In: Proceedings of the tenth international conference, Psychology of mathematics education. London 20-25 july, England

Wolters M. (1986b), Rules in arithmetic. Learning the basic facts. In: Lowenthal F. & F. Vandamme (eds.) Pragmatics and Education. Plenum Press New York

Wolters M. (1987), Schooling and the devlopment of metacognition. In: Proceedings of the second international seminar, Misconceptions and educational strategies in science and Mathematics. July 26-29, 1987 Cornell University, USA Vol. I

Zak A.Z. (1983), The development of mental acts in first to tenth graders. Voprosy Psichologii 1,43-50 (in russ.)

THE DEVELOPMENT OF THE COUNTING SCHEME OF A FIVE YEAR OLD CHILD: FROM FIGURATIVE TO OPERATIONAL

Bob Wright

Northern Rivers College

Aspects of a constructivist teaching experiment (Cobb & Steffe, 1983) involving weekly teaching sessions with four Australian children in their kindergarten year are described. The study extended the theory of children's counting types (Steffe et al., 1983) by studying children younger than those studied in an earlier teaching experiment which was the basis for the counting types theory. It also included aspects of numerical development not in the earlier study. A description, illustrated by excerpts from teaching sessions, of one child's progression from the figurative to the operational stage is given. The child creates motor, verbal, and abstract unit items when counting screened portions of collections.

Allan was one of four children in the kindergarten year of school who participated in a constructivist teaching experiment (Cobb & Steffe, 1983) during 1984. The participants were selected on the basis of an initial interview, from a kindergarten class in a school which is situated in a small regional city in New South Wales, Australia. Allan joined the teaching experiment in July 1984 and was taught approximately weekly from then until December 1984. Nine of these teaching sessions were video-taped by an assistant, and the remaining eight were audio-taped. The purpose of the teaching experiment was to extend the theory of counting types (Steffe, von Glasersfeld, Richards & Cobb, 1983) by involving younger participants in a setting culturally different from that involved in the study by Steffe and his co-workers (cf. Steffe et al., 1983; Steffe, Cobb & von Glasersfeld, 1987). This study also focussed on the role that temporal sequences of sounds and movements might play in the development of the counting scheme.

Steffe et al. (1983) identified a prdgression of five counting types; perceptual, figural, motor, verbal, and abstract; characterized by progressively less dependance on sensory input. Steffe (1984) incorporated the counting types into three stages in the construction of the counting scheme. The first is the perceptual stage, where the child can count perceptual unit items only. The second is the figurative stage in which the child counts figural, motor, or verbal unit items. Finally, at the operational stage, the child counts abstract unit items. The child at the last stage is labelled numerical and the child at the perceptual or figurative stage is labelled prenumerical.

THE FIGURATIVE STAGE

Allan's solutions of tasks in the teaching sessions of the 19 July and 3 August 1984, indicated that he had advanced beyond the perceptual stage in the construction of the counting scheme. Each of the tasks involved counting the items of a partially screened collection. In each of those two teaching sessions he counted the items of four partially screened collections. The number of screened items ranged from one to four. That Allan consistently counted the items of partially screened collections indicated that, at this time, he had advanced beyond the first counting type.

Those teaching sessions were audio-taped but not video-taped. Therefore it was not possible to determine the nature of the items that Allan created as he counted. Allan typically did not count aloud. Nevertheless a consideration of the relative times he took to count the collections indicated that he was probably counting from "one". In the same teaching sessions Allan had consistent difficulty with a second kind of task. This also involved a partially screened collection but in this case the teacher would tell Allan how many counters there were altogether and ask him to find how many were in the screened portion. The observation that Allan could not solve these tasks together with his likely counting from "one" on the first mentioned tasks indicated that he could not construct abstract unit items and therefore was in the figurative stage.

COUNTING MOTOR AND VERBAL UNIT ITEMS

In the teaching session on 14 August 1984 Allan counted the items of six partially screened collections and four collections partitioned into two screened portions. Two distinct types of counting activity were observed when Allan counted these collections. One type of counting involved tapping. The other usually did not involve sequential movements but occasionally involved nodding. Neither type involved vocal utterances or discernible lip movements. Nevertheless, as an inference from the times taken, there was little doubt that Allan counted each collection from "one". For the first task the teacher displayed five counters and directed Allan to count them. After Allan had done so the teacher screened the five counters and displayed three more. The session continued as below.

T: How many would that be altogether?

A: (Places his left hand in his mouth and makes vertical movements with his lower jaw while looking at the teacher.)

T: (After eight seconds, interprets Allan's looking at him as not understanding.) If I put those (Points at the three unscreened counters.) with those how many would that be?

A: (Looks downward. After six seconds looks up at the teacher, and smiles.) Eight.

In the last part of his solution Allan neither looked toward the counters nor made any movements. His behavior was consistent with having subvocally uttered the number words from "one" to "eight". This indicated that his number words signified countable items and therefore he counted verbal unit items. Allan was continuously engaged in counting the second collection for fifty seconds and during that period he spontaneously restarted the task three times. Allan's first attempt to count this collection is described in the following protocol. Seven counters were screened and three were visible.

T: (Screens the seven counters and then places three visible counters on the desk.) How many would that be if I put all of those together?

A: (Pauses for five seconds and then places his hands on the desk. Looks at the screen. Taps three times slowly, pauses, then taps four times slowly as before. Looks at the unscreened counters for two seconds.)

That he was apparently unable to continue counting when he looked at the unscreened counters indicates that, when he was looking at the screen and tapping, he was counting his movements rather than items which corresponded to the screened counters. His movements merely signified the screened items. Because he was focusing on his movements rather than substitutes for the screened items he was not aware that he could continue to count the visible items. This was an indication that he counted motor unit items. In this teaching session Allan also counted two collections each of which had been partitioned into two screened portions. The first of these contained two screened portions of five. Allan counted motor unit items when counting each portion of this collection. The second collection contained a screened portion of eight and one of six. Allan solution is described below.

T: (Points to the two screens in turn.) Eight, and six. How many altogether?

A: (Makes eight deliberate movements of the fingers of his right hand while looking toward the portion containing eight counters. Looks toward the screened portion of six. Makes six nods of his head, each of which involves opening and closing his mouth by holding his lower jaw and raising and lowering his head. Then looks at the teacher.) Fourteen.

When he counted the first screened portion of eight counters Allan focused on his finger movements and therefore was counting motor unit items. Although Allan nodded his head when counting the second portion it is unlikely that he counted his nods. Steffe et al. (1983) "found no evidence that ... nods ... are ever taken as countable motor items ... [and suggested that] this may be due to the fact that the kinesthetic feedback ... is automatically used by the nervous system in compensatory computation that keep's the perceiver's visual field stable" (p. 39). The most plausible suggestion is that when Allan counted the second collection his subvocal number words signified countable items and therefore he counted verbal unit items.

In the teaching session of 14 August 1984, Allan counted motor unit items on at least six occasions and verbal unit items on at least five. In the teaching sessions that followed Allan frequently counted verbal unit items but was not observed to count motor unit items. This indicated that, during the period, he was at an advanced level in his figurative stage.

FAILURE TO CREATE ABSTRACT UNIT ITEMS

In his weekly teaching sessions from 19 July 1984 onward Allan was presented with tasks which from an adult perspective would be regarded as subtractive. In one kind of task, usually referred to as missing addend, the teacher would place out a collection partitioned into a screened and an unscreened portion. The teacher would then tell Allan how many were in the whole collection and direct him to work out how many were screened. In a second kind of task the teacher would display a collection of counters and ask Allan to count them. The teacher would then ask Allan to turn away and, when he had done so, the teacher would separate and screen a portion of the collection. Allan's task was to figure out how many counters had been screened. Allan consistently failed to solve these tasks until the final teaching session on 12 December 1984. In the excerpt below, from the teaching session on 6 November 1984, it can be seen that Allan fails on a missing addend task in which five counters were visible and two were screened.

> T: (Points to the screen.) How many under here to make seven altogether? Five (Points to the unscreened counters.), and what (Points again to the screen.)?
>
> A: (Closes his eyes, looks down, and touches the backs of his hands to his forehead.) Umm, five! (Guesses.), umm.
>
> T: When I put them together I shall get seven. (after four seconds) Allan, you cover your eyes. (Removes the screen while pushing the counters together.) Okay, watch! There are seven. Now (Places the screen over the seven counters. Removes two which remain screened, while the other five counters are visible.), how many are under here to make seven?
>
> A: (After two seconds) Three!

Missing addend tasks such as these were presented to Allan in most of the teaching sessions. The solution described above, in which Allan apparently could do little more than guess was typical of his solution attempts in all but the final teaching session. Examples of the second kind of task described above were also presented in most teaching sessions. Allan also consistently failed to solve these tasks until the final session. The following example, from the teaching session on 28 November 1984, was typical. The teacher began by asking Allan to count a collection of ten counters.

T: (Passes the screen to Allan.) Cover them up with the piece of paper Allan please. (after Allan screened the counters) Now reach under and take two away. Without looking! Use your hands. (after Allan removed two counters) Now my question is how many are left (Points to the screen.).

A: Umm (Thinks for eleven seconds while looking forward and rocking in his chair.), Four! Five!

T: I shall show you again. (Pushes the two counters under the screen and then removes the screen.) How many are there now?

A: Umm (Subvocally counts the collection of ten counters.), ten!

T: (Replaces the screen and then removes the two counters as before.) How many would be there now?

A: (Thinks for seven seconds.) Six!

Until the last teaching session on 10 December 1984, Allan consistently failed to solve missing addend tasks as well as tasks similar to the one described above. This led to the conclusion that he could not create abstract unit items.

CREATING ABSTRACT UNIT ITEMS

In his final teaching session, on 10 December 1984, Allan indicated that he had advanced beyond the figurative stage. On five tasks he identified two missing addends of two counters and two missing addends of three. He failed to identify a missing addend of four counters. Allan's solutions of two of the tasks are described below.

T: (Places out eleven yellow counters.) I think there are eleven there. Can you see if I am right?

A: (Counts aloud while pointing to the yellow counters in turn.) One, two, ...eleven.

T: (Places out a screen which conceals two red counters, while leaving the eleven yellow counters unscreened.) Now, with these it makes thirteen.

A: (Looks at the screen for four seconds and then looks up at the teacher.) Two!

T: Let us see.

A: (Lifts up the screen and smiles after seeing the two red counters.)

T: Oh! Very good!

A: I know that because after eleven comes twelve and thirteen.

T: You were right! (Removes all of the counters except two of the yellow ones.) Okay, well how many are there now?

A: Two.

T: (Places out a screen which conceals three red counters, while leaving the two yellow counters unscreened.) Now with these it makes five.

A: (Looks at the screen for three seconds and then looks up at the teacher.) Three!

T: (Removes the screen to reveal the three red counters.) Very good.

Allan indicated that he could now keep track of a continuation of counting, and in so doing, identify missing addends. He had been quite unable to do this before. This counting involved creating abstract unit items.

In his final teaching session he also solved three of five tasks which involved a comparison of two screened collections. He had failed to solve similar tasks on each of four earlier occasions when these had been presented to him. One of the tasks he solved in the final teaching session (10 December 1984) involved comparing seven cubes and ten counters. Allan's solution is described below.

T: (Places out seven cubes.) Let us have that many jockeys.

A: (Looks steadily at the cubes for nine seconds and does not point.) Seven!

T: Will you cover them up! There are seven jockeys (Places out ten counters.). Tell me how many horses we have?

A: (Covers the cubes and then looks steadily at the counters for twenty-four seconds and does not point.) Ten!

T: Will you cover them up! (Slowly touches the screens in turn.) Seven jockeys, ten horses, how many horses would not have a jockey?

A: Umm (Looks at the teacher for three seconds.), three!

T: (Motions Allan to remove the screens.) You try it. Let us figure it out. Let us put the jockeys on the horses

A: (Slowly puts a cube on each counter. Then looks steadily at the counters which do not have a cube on them, and after six seconds looks up at the teacher smiling.) It was!

Allan's ability to solve tasks in his final teaching session that, in previous sessions, he responded to by guessing, indicated that a reorganization of his counting scheme was underway. This suggested that he was advancing to the operational stage in his construction of the counting scheme.

FINAL REMARKS

This paper describes one aspect of a teaching experiment which was designed to extend the theory of children's counting types (Steffe et al., 1983). It is clear from the teaching experiment that the theory can guide the teaching of five year old children who are prenumerical, and can be used to explain and predict the mathematical behavior of such children. All four children who participated in the teaching experiment made substantial progress in the construction of their counting schemes during their kindergarten year (cf. Wright, 1988). Allan, whose progress is described in this paper, was observed to construct motor and verbal unit items when counting the items of screened collections and, in his last session, he created abstract unit items on subtractive tasks.

REFERENCES

Cobb, P. & Steffe, L. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14,* 83-94.

Steffe, L. P. (1984). *Communicating mathematically with children.* Paper presented at the Fifth International Congress on Mathematical Education, Adelaide, Australia.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1987). In press.

Steffe, L. P., von Glasersfeld, E., Richards, J. & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Praeger.

Wright, R. J., (1988). *Communication in early childhood mathematics.* Paper prepared for the Sixth International Conference on Mathematical Education, Budapest, Hungary.

SAY IT'S PERFECT, THEN PRAY IT'S PERFECT: THE EARLY STAGES OF LEARNING ABOUT LOGO ANGLE.

Vicki Zack
McGill University, Montreal, Quebec

Abstract: The longitudinal naturalistic study has been investigating elementary school children's understanding of angle. Findings indicate that while discoveries about angle are indeed being made, the pace of the learning has been slow. There is a need for more time and continuity in the learning and teaching of Logo, and for more explicit teacher-elicited connections between Logo geometry and school geometry, if Logo is to play a role in the mathematics curriculum.

Within the emerging nucleus of work concerning the learning of the concept of angle in the Logo environment, there are a number of studies which devote attention to the early phases of the learning (Hillel and Erlwanger, 1983; Hillel, 1984; Noss, 1985; Hoyles, Sutherland and Evans, 1985; Kieran, 1986). This paper focuses on the learning about angle which takes place in the early stages of work with Logo-- the first 30 hours or so of Logo learning.

In my study, the exploratory "groping" stage toward the learning of angle has been more prolonged, on the part of some of the children, than I had anticipated at the outset. This longitudinal study in a naturalistic elementary school, and then secondary school, classroom computer laboratory setting has traced some of the children's work in Logo (approximately 12 sessions per year) from grade 3 (8 to 9 year-olds) to grade 7 (12 to 13 year-olds); and yet the majority of the children can be seen, in grade 7, to be still at the exploratory stage of work with angles. The difficulties experienced by the children in this study confirm some of the findings reported by the afore-mentioned researchers, but the pace of the learning has been slower. The discrepancy in pace between this study and those cited above might in part be attributed to differences in setting and curriculum agenda (for example: assigned tasks; time frame; mathematics agenda; staffing by research and mathematics experts). What is certain is that the children have more difficulty with the 'seemingly simple' aspects of Logo than the literature would sometimes have us believe.

Research was supported by a Social Sciences and Humanities Research Council Doctoral Fellowship.

In a previous paper (Zack, 1986), I presented findings concerning the level of attainment vis-à-vis angle of all of the 23 (grade 5, 10 to 11 year-old) children in my study, using as a means of focus their understanding of "right angle." This companion paper will feature two of the children from that class. It will touch upon (a) their difficulty with determining the size of a 90° and other turns; (b) the language the children use to describe turtle's location, heading, amount of turn; (c) their problem solving and recording strategies; and (d) the fact that they do not make connections between Logo geometry and school geometry.

RESEARCH DESIGN

The setting was a private, multilingual Jewish day school. All participants (11 girls, 12 boys) came from a middle class background. In 1985-86, the heterogeneous grade five class of 23 children, split into two groups, attended twelve 50-minute Logo computer sessions in the computer classroom equipped with 6 Apple IIe microcomputers, Apple Logo I software, and one printer. The expert Logo teacher, Monica Shapiro, used an individualized approach with the student pairs. The projects were child selected (exception: Monica assigned an across-class task during the last session). No changes were made to the Logo software (exceptions: use of slowturtle in grade 3; addition of a HELP command to the startup aids to help students check turtle's heading when needed). The researcher was a non-participant observer. An observational-clinical research design was used. The data included: the researcher's, the teacher's, the children's notes; interviews with the Logo teacher; three clinical interviews with the children (one at the start, one at the close of the grade five session, 1985-6, with 23 children; one at the end of the grade six year, May 1987, with 56 children); and in-depth videotape records, transcribed, of the work of five pairs of children (both the Logo work and the camera record of their interaction). Monica, the Logo teacher, wanted the children to learn the Logo theorems via exploration, and via her input when her help was solicited; but the learning of mathematics concepts via Logo was not the primary objective in her agenda.

Of the five pairs whose work was videotaped, transcribed and analyzed, I have chosen the work of one pair to discuss in this paper, that of Lilly and Rina. Their attainment in relation to the rest of the class was average; on the class grid, they will be found in the middle range of the class (Zack, 1986, p. 100).

STRIVING TO MAKE SENSE

The subtitle "Striving to make sense" pertains both to the children's trying to make sense of Logo angle, and to the researcher's trying to make sense of what the children were doing and saying vis-à-vis their turns. Lilly and Rina, and 18 of the other children in the class, used a purely visual (Kieran, Hillel, and Erlwanger, 1986) feedback strategy for determining inputs to RT/LT. I have termed it a context-referenced strategy, for they only made decisions on inputs to RT/LT when in immediate mode, using the screen as contextual reference. Rina and Lilly used their own terms, what Kieren (1987) calls ethno-mathematical language, to describe the turtle's location, and amount of turn (See below, eg. "it's straight," "go half," "go all the way around"). It was only via contextual reference to the videotape that the researcher was able to comprehend how Rina and Lilly's verbal descriptions matched the end result of the back and forth, left and right, exploratory moves that they had made.

PROBLEM SOLVING STRATEGIES AND RECORDING STRATEGIES

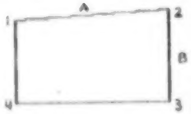
Lilly and Rina used the turtle to navigate (Sylvia Weir) around the page as they drew. Because they did not yet have a sense for the size of the rotational turn, they would "fiddle around" (Rina, May 1987) until it looked like it was perfect. They would then often say "It's perfect," and sometimes pray aloud that it be perfect.

They recorded step by step in their Hilroy book concurrently with their moves. They combined on paper by bracketing in pencil the "like" inputs. (There were no occurrences of the combining of unlike inputs, either FD X + BK Y = FD (X-Y), or, more difficult, RT X + LT Y = RT (X-Y).) Lilly and Rina used the editor as a (hopefully) accurate trace of their immediate mode commands. No debugging was done in the editor. Rina especially subscribed to a "Be safe but sure" motto concerning her Hilroy entries: "I'm not taking any chances (Dec. 4) . . . . I'll write down the mistake. As long as it turns out (Dec. 18)." In checking their Hilroy notebook inputs when attempting to find a bug, they could only resort to reconciling the number and sameness (and for Lilly, the equivalence) of the actual written entries. In trying to resolve an error in angular rotation, they were never heard to use a "Does it make sense?" test, most probably because they did not have a concept of the size of the turn against which to gauge their input.

EXPLORING THE RIGHT ANGLE TURN

I would like to consider more closely the difficulties for Rina and Lilly entailed in what might seem to an observer to be a simple task, namely that of constructing a rectangle. In grade 3 the children had been "given" RT/LT 90 for making corners. And yet when Lilly and Rina embarked upon the first part of their chosen project, the rectangle part of the tape/stereo "ghetto-blaster" they wished to make, they had to work through each corner turn. The episode below took place midway through their Logo sessions (November 20, 1985--Session #6). I chose it as a focal point of reference because it offered a glimpse at the children's moves, their use of language, their awareness of visual cues which signal error, and their interaction with neighbouring peers vis-à-vis their product.

They arrived at the first turn (marked 1 below) by keying RT 35, RT 35, RT 10, RT 5. Lilly declared: "It's straight." As soon as they proceeded with Line A, they saw that the line was jagged. Their evaluation followed. Lilly said, "It's good"; Rina said, "It's bad"; Lilly countered with: "It doesn't matter." [However, as became clear in subsequent comments, the jagged line did bother them--Rina especially-- very much.] At this point they let it be. Rina expressed surprise at the combined sum--85, and stopped to reconfirm with Lilly that it did indeed take "RT eighty-five" to "get all the way around."



For turn 2, Rina used the information from the previous turn. She stated: "We wanna go half, we want RT 85." [It is only by contextual reference to the screen, and by the fact that one knows that they are aiming for a "corner" that one follows that "all the way around" and "we wanna go half" both refer to a quarter turn.] Rina and Lilly then proceeded to disagree about the input, and it sounded as if they were still disagreeing when they both decided on an input of RT 95, which, with serendipity, was the correct input. The subsequent line, Line B, was straight.

Turns 3 and 4 were the results of inputs of 90; there were lateralisation errors, but both children agreed that 90 was the input, and they used corrections of 180 when needed. One might think that they had now grasped the importance of 90 in making a smooth corner with straight line arms. It was clear however that they had not yet mastered the 90 when one listened to Rina and Lilly's interaction with neighbouring peers (Russ

Michael) which followed shortly after the teacher had come by and "ticked off" that the rectangle part was done.

Russ: Rina, Rina- Shouldn't the line on top be straight?
Rina: Ya, but we didn't do that.
Russ: So why didn't ya do it?
Rina: We didn't do it. (Points to the CRT) It turned out like that.
Russ: Ya, 'cause you did something wrong.
Rina: No, we didn't do 'anything wrong.
Lilly: 'No, we didn't.
Russ: (inaudible)
Rina: (getting back to work, leafing through the pages of her recording book)
'Kay, we have to get into the editor.
Michael: You DID, 'cause you went downwards. (inaudible)
Russ: A line would never be like that.
Rina: You wanna bet it would? It happened '(? to me a couple o' times)
Lilly: 'BIG DEAL! (Now sitting straight, looks at her book)
Rina: O.K. Logo editor. (Looks away from Russ and Michael, signalling the end of their interchange with the boys.)

The jagged line served as a cue to Russ and Michael, as it had served to Lilly and Rina, that something was wrong. But the girls did not respond well to the peer intervention by Russ and Michael. Rina and Lilly's lukewarm reception of Russ and Michael's comments may be due to the fact that (1) the boys were offering an unsolicited,negative evaluation of their product; and (2) Lilly and Rina had just completed the "rectangle" part of their project and were anxious to get on with the next part of their work.

I wondered when reviewing the tapes whether Lilly and Rina had desisted from debugging because they were rushed, or because they were unable to correct. I therefore looked at subsequent tapes closely and noted evidence as late as January 29, 1986 that Lilly was not completely in command of the 90 as input. In the January 29 teacher-assigned across-class task requiring squares, Lilly groped for the input to LT (LT 50, LT 19, LT 11), then suddenly cleared the screen and said sharply: "LT 90!" During the clinical interviews in Feb. 1986, and in May, 1987, I asked Lilly and Rina about the rectangle. When asked (in Feb. 1986) what she could have done differently in making the rectangle, Lilly was able to state that she would have had to turn RT 90 "to make (the line) straight."

Rina, however, in both the February 1986 interview, and in the May 1987 interview, asserted that she did not know how she could have made "that corner even. . . . I don't know why that (i.e. the jagged line) happens."

During the February 1986 (Zack, 1986) and the May 1987 interviews, it was clear that Rina and Lilly had made progress in their understanding of certain aspects of angular rotation, though there were still gaps to be filled. Rina was able to identify a right angle in different orientations (on paper). She was able to use it as a point of reference when needed. For example, in proving that the blackboard angle must be obtuse, she showed how it was more than 90; 14/56 students were seen to use 90 as a reference in this way. Rina was not able to use an analytic (Hillel, Kieran and Erlwanger, 1986) problem-solving approach to determine the supplement for an angle of 175. She stated that she "would fiddle around" until she got where she wanted to be. She was not able to use analytically the classroom geometry information she knew by rote, namely that there are 180° in a straight line. She was also still working toward consolidating the fact that the input to RT/LT is equal to the number of degrees in an angle (Zack, 1986).

Lilly was able, with prompting, to figure out the amount of turn needed for the supplement of a 50° turn. [A total of 10/56 or 17.8% of the students interviewed were able to use an analytic approach with prompting; and 5/56 or 8.9% of the students were able to do so without prompting.] Lilly used a method employed by three of the students who were able to use an analytic approach. She first moved the cardboard turtle through a turn of 90; and then worked within the one remaining quadrant, determining that the complement of the given angle was 40.

IMPLICATIONS FOR THE CURRICULUM--NOW WHAT?

The pace may seem slow, but the learning offers a rich foundation upon which to build. For Lilly and Rina, the right angle and its relationship to other angles would be a fruitful starting point, in view of the time they spent making some sense of it. The findings indicated that Rina and Lilly, and the majority of the children interviewed, had not been able to make connections between Logo geometry and school geometry (Zack, 1986), nor to work analytically within the Logo environment itself; indeed, the fact that they were "drawing" precluded the need to do so. And yet as the project drew to a close in January, 1988, it was clear that major obstacles lay in the way of future progess in the children's learning of angle via Logo, namely: (1) the children's perception about Logo;

(2) the time frame; and (3) WHO was going to help make the connections. The perception of some of the children interviewed in May 1987 was that Logo was "just drawing" (13/54, or 24%); the majority felt that they had learnt everything there was to know, and that they had outgrown Logo by grade 5 or 6. When asked "Do you feel that Logo can help you learn about angles? How?", eleven out of the fifty-six children interviewed, including Lilly, said that they never thought of angles while doing Logo. But certainly a teacher could highlight the vital connections. The question then is when and who. It was the elementary Logo teacher's (Monica's) expectation that the high school curriculum would be the most appropriate stage for the explicit invoking of connections. At the end of the grade 7 Logo component (again approximately 10 sessions in November to January, 1987-88, and the last year of Logo), it was clear that Monica's expectation was not going to be realized. The reasons were, briefly, as follows. The computer teacher spoke of the limited time span he could allot to Logo within the grade seven computer curriculum, he noted the pupil-teacher ratio (1/24), and the desire to cover topics such as "variables" and programming skills. The computer teacher (who had also taught the mathematics grade 7 course) overestimated what the children knew about angles and Logo geometry. The two Grade 7 mathematics teachers interviewed stated that they did not make any connections to Logo geometry in their mathematics classroom; one said this was due in part to her unawareness of what the children had learnt.

If the slow pace of the learning detected in this study reflects the pace in many current Logo school programs, this factor may in part explain the frustration with Logo that Watt and Watt (1987a) have reported teachers are feeling. The Watts (1987b) have cited teachers' complaints that "Logo isn't working" and that "students aren't learning important mathematics and computer science through exploration and discovery." We are in some danger, then, of throwing out the proverbial "baby with the bath water." Findings from this study indicate that there is a need for a more concentrated, continuous time frame for the Logo learning, a need for an underlying but not restricting mathematics agenda, and a need for teacher-elicited explicit connections to be made concerning both the underlying structures of Logo geometry, and the relationship between Logo geometry and school geometry, if Logo is to play the contributing role it can play in the mathematics curriculum.

REFERENCES

Hillel, J. (1984). *Mathematical and programming concepts acquired by children, aged 8-9, in a restricted Logo environment.* (Research Report No. 1). Department of Mathematics, Concordia University, Montreal, Quebec.

Hillel, J., & Erlwanger, S. (1983). *Observations, reflections and questions about children's Logo learning.* Unpublished paper, Concordia University, Montreal. Quebec.

Hoyles, C., Sutherland, R., & Evans, J. (1985). *A preliminary investigation of the pupil-centered approach to the learning of Logo in the secondary school mathematics classroom.* Logo Maths Project, University of London Institute of Education.

Kieran, C. (1986). Logo and the notion of angle among fourth and sixth grade children. *Proceedings of the Tenth Annual Meeting of Psychology of Mathematics Education (PME 10),* July 20-25, 1986, London, England, 99-104.

Kieran, C., Hillel, J., & Erlwanger, S. (1986). Perceptual and analytical schemas in solving structured turtle-geometry tasks. *Proceedings of the Second International Conference for Logo and Mathematics Education,* July 15-18, 1986, London, England, 154-161.

Kieren, T. (1987, January). *Language use in mathematics.* Visiting lecture, January 26, 1987, Concordia University Department of Mathematics, Montreal, Quebec.

Noss, R. (1985). *Creating a mathematical environment through programming: A study of young children learning Logo.* Published doctoral dissertation, University of London Institute of Education.

Watt, D. H., & Watt, M. L. (1987a). *Assessing Logo learning in classrooms.* Talk presented at the Fifth Annual Conference on Computers in Education, McGill University, Montreal, Quebec, November 18-20, 1987.

Watt, M. L., & Watt, D. H. (1987b). *Exploratory research on critical aspects of Logo learning: A progress report.* Education Development Center, 55 Chapel Street, Newton, Massachussetts 02160.

Zack, V. (1986). A study of ten to eleven year-old students' notion of right angle in Logo geometry. *Proceedings of the Second International Conference for Logo and Mathematics Education,* July 15-18, 1986, University of London Institute of Education, 96-103.

SUBSTITUTIONS LEADING TO REASONING

Nurit Zehavi

The Weizmann Institute of Science, Israel

A software package which combines skill and reasoning for substitution in algebraic expressions was developed in the Department of Science Teaching at the Weizmann Institute. A study based on the implementation of the software was conducted (n=85), and 6 teachers were involved in cognitive workshops. The workshops incorporated a psychometric method that applies an index called the Caution Index, which detects unusual response patterns. The research instruments were a test which required a combination of skill and logical reasoning in substitution tasks, and the last program in the software (a game). The test's results can be related and explained by the kinds of effect of the software on various types of student. Observation of students playing the game and evaluating their achievement of the learning goals led to patterns for adaptive implementation of the software for individual students.

The problems and difficulties which students have in algebra have been the subject of much investigation (Rosnick and Clement, 1980; Matz, 1981). According to Wheeler & Lee (1986), the algebra school curriculum forces pedagogy to oscillate inconsistently between presenting algebra as a universal arithmetic and as a formal symbolic system. This affects student conception of justification in algebra; for example, a single numerical substitution can lead to incorrect reasoning and the "justification" for changing an algebraic equation.

In the traditional repertoire of activities in the junior high school algebra curriculum, the student is mainly concerned with manipulation of expressions, word problems and solving equations and inequalities. The introduction of microcomputers in the classroom enables the design of novel activities which may help to bridge the arithmetics/formal symbolic divide. These activities fall into three main types: the learning and practice of algebraic skills as part of a strategic environment, algebraic tasks involving programming, and microworlds that provide access to multiply linked representations.

Kaput (1986) believes that novel software environments will help shape the direction of mathematics teaching and learning, if reasonable teacher training support is given. On the other hand, he mentions (in relation

---

The author is grateful to Naomi Taizi and Nira Schwarzberg for their contribution to the development of the software and to the study. Thanks to Prof. M. Bruckheimer for his contribution to the consolidation of the ideas presented in this paper.

to an example of a software in algebra) that we are a long way from understanding how to exploit this new tool pedagogically.

The Department of Science Teaching at the Weizmann Institute maintains a curriculum project in mathematics which integrates educational cognitive research with the practicel activities of development and implementation. This integration applies equally to the development of software within the curriculum project (Zehavi et al, 1987; Zehavi, in press). In this paper we describe a study based on the implementation of a software package of the first type above with focus on cognitive workshops with teachers as part of the guidance process.

The workshops incorporated a psychometric method that applies an index called the "Caution Index", which detects unusual response patterns and is obtained from a student-problem curve developed in Japan by Sato (see Tatsouka, 1984; Birenbaum, 1986). A binary data matrix is suitably rearranged so that an unusual response pattern for either an item or a student can easily be identified. The anomalities expressed by the caution indices can be related and explained by the kinds of effect of the software on various types of student.

THE SOFTWARE

The rationale in the development of the software is that we want to offer activities which combine skill and reasoning for substitution in algebraic expressions. The activities involve one-dimensional dynamic presentation of the role of parameters in algebraic expressions. The software contains two tutorial units and two competitive games.

The basic task is to separate a list of increasing numbers, according to the sign of the result of their substitution into a given expression. At the beginning the tasks involve expressions, for which there is only one change of sign:

$$b(tx + a) \quad \text{or} \quad \frac{b}{tx + a}$$

In Figure 1 the numbers to the left of the dividing stroke give negative results when substituted in $x - 7$ and the numbers to the right give positive results.

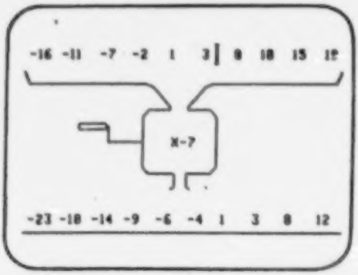


Figure 1: Substitution in x - 7.

A game for two, "Warring Expressions", offers a strategic environment which requires mathematical logical reasoning in addition to skill-drill. Each player gets a random list of numbers, which remains throughout the game and an open phrase which changes at each turn (see Figure 2). The aim of each player is to be the first one to "turn on" all the numbers in his/her list. To achieve this, at each turn, a player can choose to "turn on" numbers in his list that give positive results, or "turn off" numbers that give negative results in the opponent's list.

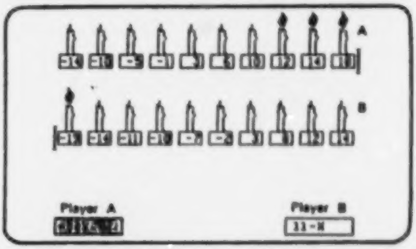


Figure 2: Warring Expressions.

To illustrate the skills and reasoning which are involved, we consider the situation in Figure 2. It is player B's turn. If (s)he chooses list B, the divider should be moved to the right and placed between -10 and -7, lighting of the numbers to the left. (If a player places the divider incorrectly (s)he loses the turn. This may possibly happen here because of difficulties in dealing with the double negative, in the list and in the expression -9-x.) If player B chooses list A, in order to cause his/her opponent trouble, the divider should be placed between -10 and -5, and then the three numbers to the right will be "turned off". Note that at each turn a player can (and should) consider the other player's expression. In the example, if player B does not stop player A, the latter can win the game in the following move.

The next tutorial unit deals with expressions which have two changes of sign:

$$(\pm x + a)(x + b) \quad \text{or} \quad \frac{\pm x + a}{\pm x + b}$$

The following game, "The Expression Strikes Back", requires a higher level combination of skill and reasoning. This game and its role in the study will be described later.

THE STUDY

Three Grage 8 classes in one junior high school participated in the study. One of them was the experimental class and the two other classes formed the control group. The three classes were of about the same average ability as measured by an achievement test administered by the school teachers:

Experimental class (n=28): mean score 75.7, standard deviation 13.4.

Control classes (n=57): mean score 74.6, standard deviation 14.2.

The software was presented to the expertimental class after the students were taught the techniques for solving linear equations and inequalities. They worked on the three first programs (the first tutorial, the first game and the second tutorial) for three lesson periods. The study was conducted as part of an in-school cooperative guidance system and thus the teachers of the three classes and three student-teachers observed the students using the software.

A test which required combination of skill and logical reasoning in substitution tasks was given to all three classes. Student responses were checked by the researchers and the Sato statistical method was applied. The findings were described and discussed with the teachers in cognitive workshops.

Test results and discussion

Student scores for the experimental class had a correlation of 0.73 with the school-achievement test scores. As expected higher correlation was found for the control classes since no treatment was given.

The substitution test contains four parts. In the following we bring the results for the two groups on the last two parts. An asterisk (*) is used to indicate an item for which the caution index (CI) was found to be larger than 0.3. This indicates the existence of an anomality in the response pattern; that is, some low scorers on the test answered that item correctly and some high scorers missed it.

The third part requires high level combination of skill and reasoning, regular techniques do not help. The items were presented in ascending order of complexity as can be seen from the results.

Group Item	Experimental n=28	Control n=57
9	86	65*
10	75	51*
11	68*	39
12	54*	11
13	50	8
14	43	0

Table 1: Results for Part III

The difference in favor of the experimental class, probably due to the effect of the software, is very clear. We found high caution indices for Items 9 and 10 in the control group which means that some low scorers did not stick to techniques and reasoned correctly. Items 11 and 12 are of interest for the experimental group.

Item 11: $\{x \mid x < 2\}$ is the truth set for $4(x + \boxed{\ }) < 0$.

Fill in the blank.

Item 12: $\{x \mid x \bigcirc 3\}$ is the truth set for $-2(x + \boxed{\ }) > 0$

Fill in the blanks.

These two items are the only ones in this part that involve an expression of the form $(x + \boxed{\ })$, where the blank has to be filled by a negative number. In the software, students had a chance to practice with expressions of this sort and we observed difficulties. It seems that the feedback given to some generally low achieving students caused their awareness of such situations.

The last part deals with quadratic expressions, and the students were also asked to generalize their answer.

For example, Item 17:

The number -3 belongs to the truth set of $(x + 6)(x + \square) > 0$.
Which numbers can fill the blank?
The results are given in the table.

Group	Experimental	Control
Item	n=28	n=57
15	96	81
16	86	61
17	75*	39
18	61	39
19	64*	37
20	32	9

Table 2: Results for Part IV

Here again we can compare the responses of the two groups and discern the possible effects of the software. From the student-problem data matrix, we can also detect the students who seem to benefit more than others.

Let's look at two possible solutions of Item 17: One can argue as follows: since $-3 + 6$ is positive, the second factor must be positive, and $-3 + \square$ is positive for $\square > 3$. Another way is to substitute -3 and obtain a numerical inequality $(-3 + 6)(-3 + \square) > 0$, simplify to obtain $-9 + 3 \cdot \square > 0$ and then solve for $\square$. The software provides opportunities for arguments such as those in the first solution. In fact, 21 students out of 28 gave a correct answer using such arguments. Some of them were low scorers on the whole test (CI=0.43). Among the 7 students who made mistakes, 2 used the second method. In the control group 14 students (out of 57) used the second method, of which, some solved it correctly and others made mistakes.

We notice that Item 17 includes an expression of the type $(x + \square)$ as in other items with anomalies in the response pattern. Item 18 was similar but more difficult with a low caution index (0.12), it involved an expression of the type $(x - \square)$.

Creative observation

In the cognitive workshop the teachers considered the structure of the software in depth. They were now ready to observe individual students playing the second game, The Expression Strikes Back (see Figure 3). A brief description of the game follows.

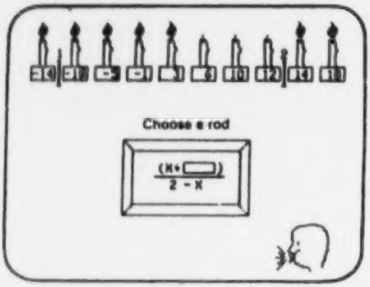


Figure 3: The Expression Strikes Back

There is one list of numbers, two dividers and an expression or the four $(a \pm x)$ $(x + \square)$, or $\frac{x + \square}{a \pm x}$. Numbers are "turned on" if they give a positive outcome when substituted in the current expression, and turned off if the outcome is negative. The first player aims to "turn on" all the numbers in the list and the second aims to turn them off. In Figure 3, it is the turn of the second player, who has to choose one divider, move it and then fill in the blank so that the dividers separate those numbers that give a positive result when substituted in expression from those which give a negative result. In this case, if the player reasons correctly, (s)he will prefer to move divider 1 to the right of the number -1 and write -13 in the blank. The "computer" will then turn off the lights, but the light above the number 3. In the design of the game we had two intentions. To provide opportunity to crystallize and generalize the tasks of the first three programs, so that the learner will achieve the goals of the software. At the same time, we wanted to be able to evaluate student actions. Therefore, we designed it in such a way that it is, in fact, free of strategic considerations (which creates "noise" in the evaluation process). However, since the tasks and the rules are quite complex, the game attracts students before they have gained mastery and is thus a learning environment.

We chose six students whose achievement differed as measured by the substitution test and who had different caution indices, thus representing various response patterns. We asked the six teachers and student-teachers to play the game individually with the students. The teachers were instructed to make the least move, with no explanation and to record and assess the student's actions. Based on their observation they suggested patterns for effective implementation of the software including related worksheets for individual students. Some of the observation protocols and teacher suggestions will be presented in the talk.

CONCLUDING REMARKS

There exist several attempts to use computer software in the teaching of investigation of algebraic expressions by using graphs of function. Such presentation requires formal interaction of algebraic and graphical concepts. Our experience with junior high school algebra teaching indicates that informal one-dimensional presentation could serve as a preparatory stage. This was our starting point in the development of the software described above. The idea was to enhance student ability to combine skill and reasoning.

Another aspect of the study was teacher involvement in the evaluation and adaptation system. A repetition of the method in some more schools will help us to formalize diagnostic patterns for effective implementation which will be used in the further development of flexible adaptive versions of the software.

REFERENCES

Birenbaum, M. (1986). now" - beyond the "What", towards the "Why": A rule-assessment approach to achievement testing. Studies in Educational Evaluation, 12, 159-168.

Kaput, J.J. (1986). Information Technology and Mathematics: opening new representational windows. The Journal of Mathematical Behavior, 5, 187-208.

Matz, M. (1981). Toward a process model of high school algebra errors. In D.H. Sleeman & J.S. Brown (Eds.). Intelligent Tutoring Systems. London: Academic Press.

Rosnick, P. & Clement, J. (1980). Learning without understanding: the effect of tutoring strategies on algebra misconceptions. Journal of Mathematical Behaviour, 3, 3-27.

Tatsuoka, K. (1984). Caution indices based on item response theory. Psychometrika, 49, 95-110.

Wheeler, D. & Lee, L. (1986). Towards a psychology of algebra. Proceeding of the Eight Annual Meeting of the NA Chapter of the International Group for the Psychology of Mathematics Education. (pp. 133-138). East Lansing, Michigan.

Zehavi, N., Gonen, R., Omer, S. & Taizi, N. (1987). The effects of microcomputer software on intuitive understanding of graphs of quantitative relationship. Proceedings of the Eleventh Conference for the Psychology of Mathematics Education, pp. 255-261, Montreal.

Zehavi, N. (In press). The development of mathematics software: an interactive process. The Journal of Computers in Mathematics and Science Teaching.

LIST OF AUTHORS

Ainley, Janet
University of Warwick,
Dept.of Sci.Education,
Coventry CV4 7AL
GREAT BRITAIN

Albert, Jeanne
Weizmann Institute of
Science, Dept.of Science
Teaching,
Rehovot 76100
ISRAEL

Alibert, Daniel
CNRS, Institut Fourier,
B.P. 74
38402 Saint-Martin-D'Heres
Cedex
FRANCE

Alston, Alice
Rutgers University,
Center for Math.,
Science and Computer
Education
S.E.R.C. Room 239
Busch Camp.
New Brunswick,
New Jersey 08903
USA

Amit, Miriam
Ben-Gurion Univ.
Dept.of Match.
Beer-Sheva, POB 653
84965
ISRAEL

Becker, Gerhard
University of Bremen
Modersohnweg 25
D-2800 Bremen 33
GFR

Behr, Merlyn
Dept.of Mathematical
Sciences Northern
Illinois Univ.
DeKalb
IL 60115
USA

Bliss, Joan
King's College London
Center for Education
Studies
552 King's Rd.
London SW10 0VA
GREAT BRITAIN

Bell,Garry
Northern Rivers College
of Advanced Education
Box 157,
Lismore,
AUSTRALIA
2480

Bergeron, Jacques C.
Uninversité de Montréal
Fac.des Sciences de
l'Éducation;Succ."A"
C.P. 6128
Montréal, Québec
CANADA
H3C 3J7

Bishop, Alan J.
Cambridge Univ.
Dept.of Education,
17 Trumpington Str.
Cambridge CB2 1QA
GREAT BRITAIN

Boero, Paolo
Dipartimento di Matematica
dell'Università
Via L.B.Alberti 4,
16132 Genova,
ITALIA

Boulton-Lewis,
Gillian Marie
Brisbane College of
Advanced Education,
130 Victoria Park Road,
Kelvin Grove.
Queensland,
AUSTRALIA
4059

Brandau, Linda
Univ.of Calgary
Dept.of Curriculum and
Instruction,
Calgary, Alberta
CANADA
T2N 1N4

Brink, Jan van den
Vakgroep OW et OC,
Univ.of Utrecht,
Tiberdreef 4
3561 GG Utrecht
THE NETHERLANDS

Bromme, Rainer
Institut für Didaktik der
Mathematik,
Univer.Bielefeld
Postfach 8640
D-4800 Bielefeld 1.
GFR

Bruckheimer, Maxim
Weimmann Institute,
Science Teaching Dept.,
Rehovot
76100
ISRAEL

Carraher, Terezinha Nunes
Ministerio da Educaçao
Universidade Federal de
Pernambuco CFCH 8°
Andar Cidade Universitaria
50 000
Recife- PE
BRAZIL

Chazan, Daniel
Harvard Educational Technology
Center
337 Gutman,
Appian Way
Cambridge MA 02138
USA

Clements,M.A./Ken/
Deakin Univ.,
School of Education,
Geelong,
Victoria 3217
AUSTRALIA

Cohors-Fresenborg,
Elmar
Universität Osnabrück
FB Math./Informatik,
Postfach 44 69,
D-4500 Osnabrück
GFR

Cooper, Martin
Univ.of New South Wales
School of Education
P.O.Box 1
Kensington, N.S.W.
AUSTRALIA
2033

Crawford, Kathryn
Sydney Univ.,
Fac.of Education,
Sydney N.S.W.
AUSTRALIA
2006

Davis, P. Joy
Open Univ.,
Centre for Maths
Education
Math.Fac.,
Milton Keynes MK7 6AA
GREAT BRITAIN

De Corte, Eric
University of Leuven,
Center for Instructional
Psych.,
Vesaliusstraat 2,
B-3000 Leuven,
BELGIUM

Drouhard, Jean-Philippe
UER de Didactique des
Disciplines Univ.
Paris 7
45-46/1o6 Univ.
Paris 7 2Pl
Jussieu
75251 Paris
Cedex 5
FRANCE

Dubinsky, Ed
Purdue Univ.,
Education Dept.,
Lafayette Indiana
479o7
USA

Edwards, Laurie D.
Univ.of California,
1921 Francisco St 4A
Berkeley CA 947oo
USA

El-Faramawy, Hamdy A.
Menoufia Univ.
Shebin El-Kom,
EGYPT

Ellerton, Nerida F.
Deakin Univ.,
School of Education,
Geelong,
Victoria 3217
AUSTRALIA

Ernest, Paul
Univ.of Exeter School
of Education;
St,Lukes, Heavitree Road,
Exeter, Devon
EX1 2LU
GREAT BRITAIN

Evans, Jeffrey T.
Middlesex Polytechnic;
1 Granville Rd.
London N.4 4EJ
GREAT BRITAIN

Even, Ruhama
Michigan State Univ.,
Math. Dept.,
Wells Hall, MSU
East Lansing, MI, 48824
USA

Feyu, Cao
Curriculum and Teaching
Materials Research Inst.
55 Sha Tan Hou Street,
Beijing
CHINA

Fresko, Barbara
Weizmann Institute of Sci.,
Dept. of Science Teaching,
Rehovot 761oo
ISRAEL

Friedlander, Alex
Weizmann Institute of Sci.,
Dept. of Science Teaching,
Rehovot 761oo
ISRAEL

Fou-Lai Lin
Math.Dept., National Taiwan
Normal Univ.;
Wolfson College,
Cambridge CB3 9BB
GREAT BRITAIN

Fry, Carol J.
The Ohio State Univ.,
2o2 Arps Hall,
1945 North High Street
Columbus,
Ohio 43210
USA

Fujii, Toshiakira
Univ. of Tsukuba,
Inst. of Education,
Sakura-Mura,
Tsukuba-Shi,
JAPAN
3o5

Gattuso, Linda
LÉGEP Du Vieux-Montréal,
5225 Kensington Ave,
Montréal H3X 3S7
CANADA

Genzwein, Ferenc
Hungarian National Centre
for Educational Technology
/OOK/
15o2 Budapest
POB 26o
HUNGARY

Goldin, Gerald A.
Rutgers Univ.,
Center for Math.,
Sci. and Computer Education
Sero BLDG. RM. 239,
Busch Campus,
New Brunswick,
N.J. 08903
USA

Gurtner, J.-L.
University de Fribourg

Habermann, Gusztáv M.
"Juhász Gyula" Teacher Training
College /Juhász Gyula Tanárképző Főiskola/
Szeged 67o1
Április 4 utja 6.
HUNGARY

Halevi, Tirza
Weizmann Institute,
Science Teaching Dept.,
Rehovot 76100
ISRAEL

Halmos, Mária
Hungarian National Centre for
Educational Technology /OOK/
15o2 Budapest, POB 26o
HUNGARY

Harel, Guershon
Dept.of Math.Sciences,
Northern Illinois Univ.
DeKalb,
IL 60115
USA

Hart, Katherine M.
Kings College, Nuffield
Secondary Mathematics,
552 Kings Rd.,
London SW1o OUA
GREAT BRITAIN

Herscovics, Nicolas
Concordia Univ.;
3838 Eduard-Montpetit Blvd.
Montréal, P.Q.
CANADA
H3T 1K9

Hershkowitz, Rina
Weizmann Institute,
Science Teaching Dept.
Rehovot 76100
ISRAEL

Hillel, Joel
Concordia Univ.,
Dept. of Math. HB-234,
7141 Sherbrooke St. W.
Montréal, Québec H4B 1R6
CANADA

Hirabayashi, Ichiei
Nara Univ. of Education,
Takabatake-cho, Nara City
63o
JAPAN

Hoyles, Celia
University of London,
Inst.of Education,
2o Bedford Way,
London WC1H OAL,
GREAT BRITAIN

Jaworski, Barbara
Open Univ.,
Centre for Maths
Education,
Walton Hall,
Milton Keynes MK7 6AA,
GREAT BRITAIN

Kieran, Carolyn
Université du Québec
Montréal
Dept. Math./Info.;
C.P. 8888, SUCC. "A",
Montréal, Québec,
CANADA
H3C 3P8

Klein, Sándor
"Juhász Gyula" Teacher
Training College
/Juhász Gyula Tanárképző
Főiskola/
Szeged 67o1
Április 4 utja 6,
HUNGARY

Kurth, Wilfried
Universität Osnabrück
Postfach 4469
D-4500 Osnabrück,
FRG

Kuyper, Hans
RION Univ. of Groningen
RION
Postbus 1286,
97o1 BG. Groningen
THE NETHERLANDS

Kynigos, Chronis
University of London
Inst. of Education,M.S.C.
Dept.
2o Bedford Way,
London N.C.1.
GREAT BRITAIN

Lacasse, Raynald
Univ.D'Ottawa
651 Cumberland Ont.
CANADA
K1N 6N5

Lea, Hilda
Univ. of Botswana;
Private Bag 22,
Gaborone,
BOTSWANA

Lean, G.A.
Deakin Univ.,
School of Education,
Geelong, Victoria 3217
AUSTRALIA

Lemerise, Tamara
U.Q.A.M. Département de
psych. C.P. 8888,SUCC."A",
Montréal, QC., H3C 3P8
CANADA

Linchevski, Liora
Hebrew Univ.,
School of Education,
43 Shmaryahu Levin,
Jerusalem 96664
ISRAEL

Lowenthal, F.
Université de l'État a Mons.
Laboratoire N.V.C.D.
2o, Place du Parc
7000 Mons
BELGIUM

Lymberopoulos-Fioravantes,
Hélène
UER de Didactiques des
Disciplines Université
Paris 7;
UER de Didactique 45-46/1o6
Univ.
Paris 7 2P1
Jussieu 75251 Paris Cedex 5
FRANCE

Maher, Carolyn A.
Rutgers Univ.,
Center for Math.,
Sci. and Computer
Education S.E.R.C.
BLDG. Room 239 Bush Campus
New Brunswick,
New Jersey 08903
USA

Mason, John H.
Open Univ.,
Centre for Maths Education,
Math. Fac.,
Milton Keynes MK7 6AA
GREAT BRITAIN

Mevarech, Zemira
Bar-Ilan Univ.,
School of Education,
Ramat-Gan,
ISRAEL

Mulligan, Joanne
Catholic College of Education
Sydney, 11 Cobham St.
Maroubra
AUSTRALIA

Nesher, Perla
Univ. of Haifa,
School of Education,
Mount Carmel
Haifa 31999
ISRAEL

Nikolakarou, Hélène
UER de Didactique des
Disciplines Univ.
Paris 7,
UER de Didactique 45-46/106
Univ. Paris 7 2Pl. Jussieu
75251 Paris Cedex 5
FRANCE

Nohda, Nabuhiko
Inst. of Education,
University of Tsukuba;
Takezono 2-809-1
Tukuba-city,
Ibaraki-ken, 305
JAPAN

Noss, Richard
Univ. of London,
Inst. of Education,
20 Bedford Way,
London WC1H 0AL
GREAT BRITAIN

Olivier, Alwyn
Univ. of Stellenbosch,
Dept. of Didactics,
Fac. of Education,
Stellenbosch, 7600
SOUTH AFRICA

Otten, Wilma
RION Univ. of Groningen,
RION / Postbus 1286
9701 BG. Groningen,
THE NETHERLANDS

Paquelier , Yves
UER de Didactique des
Disciplines Univ.Paris 7,
UER de Didactique 45-46/106
Univ. Paris 7, 2Pl
Jussieu 75251 Paris Cedex 5
FRANCE

Post, Thomas
Univ.of Minnesota,
240 Peik Hall,
Minnesota, MN 55455
USA

Robert, Aline
CNRS France
Paris V - Labo PSYDEE
46 rue Saint-Jacques
75oo 5 Paris
FRANCE

Rogalski, Janine
CNRS France
Paris V - Labo PSYDEE
46 rue Saint-Jacques
75oo 5 Paris
FRANCE

Rogers, Pat
York University, Math.Dept.
47oo Keele Street,
North York, Ont.,
M3J 1P3
CANADA

Sakonidis, H.N.
King's College London,
Center for Educational
Studies,
552 King's Rd.
London SW10 OUA
GREAT BRITAIN

Schliemann, Analucia Dias
Universidade Federal de
Permanbuco
Rua Arao Lins de Andrade,
275 - Piedade Jaboatao - Pe
BRAZIL

Schwarz, Baruch
Weizmann Institute,
Science Teaching Dept.,
Rehovot 761oo
ISRAEL

Sfard, Anna
Hebrew University,
The Israeli Sci. Teaching
Center, Givat-Ram,
Jerusalem 919o4
ISRAEL

Shanghe, Cai
Curriculum and Teaching
Materials Research Inst.,
55 Sha Tan Hou Street,
Beijing.
CHINA

Shigematsu, Keiichi
Nara Univ. of Education,
Takabatake-cho,
Nara-City, 63o
JAPAN

Sierpinska, Anna
Inst.of Math.,
Polish Acad. of Sci.;
IM PAN, P.B. 137
00-950 Warsaw
POLAND

Simon, Martin A.
Mount Holyoke College;
Symmermath for Teachers,
MT. Holyoke College,
South Hadley, MA, 01075
USA

Sinkinson, Anne
Kings College, London,
Nuffield Secondary Math.
552 Kings Rd.,
London SW10 OUA
GREAT BRITAIN

Southwell, Beth
Nepean College of
Advanced Education;
31 Dorothy Street Ryde,
N.S.W. 2112
AUSTRALIA

Steinbring, Heinz
Inst. für Didaktik der
Math. Univer. Bielefeld
Postfach 864o
D-48oo Bielefeld 1
FRG

Streefland, Leen
Vakgroet OW and OC
Univ. of Utrecht
Tiberdreef 4
2561 GG Utrecht,
THE NETHERLANDS

Tall, David
Warwick University
Maths. Education
Res. Centre,
Coventry School -
Bablake,
Coundon Road
Coventry CV1 4AU
GREAT BRITAIN

Thomas, Michel O.J.
Warwich University
Maths. Education
Res.Centre,
Coventry
School - Bablake
Coundon Road
Coventry CV 1 4AU
GREAT BRITAIN

Tompa, Klára
Hungarian National
Centre for Education
Technology /OOK/
15o2 Budapest
POB 26o
HUNGARY

Van Coillie, V.
Univ. of Leuven,
Center for Instructional
Psych.,
Vesaliusstraat 2,
B-3000 Leuven
BELGIUM

Verschaffel, Lieven
Univ. of Leuven,
Center for Instructional
Psych., Vesaliusstraat 2,
B-3000 Leuven
BELGIUM

Vinner, Shlomo
Hebrew Univ.
School of Education;
43 Shmaryahu Levin
Jerusalem 96664
ISRAEL

Visser, Delene
Univ. of South Africa,
Dept. of Psych.
POB 392
Pretoria, SOUTH AFRICA

Wolters, Miriam A.
Univers. of Utrecht.,
Dept. of Developmental Psych.,
Heidelberglaan 2,
3584 CS Utrecht,
THE NETHERLANDS

Wright, Bob
Northern Rivers College,
POB. 157
Lismore, N.S.W.
AUSTRALIA
248o

Zack, Vicki
McGill Univ., Fac. of Education
5822 Einstein Ave,
Montréal, QC
CANADA
H4W 2Y6

Zehavi, Nurit
Weizmann Institute,
Science Teaching Dept.,
Rehovot 76100
ISRAEL

PME XII

BUDAPEST
VESZPRÉM

HUNGARY

Local Organizing Committee:
Hungarian National Centre for Educational Technology (OOK)
Mailing address H-1502 POB 260 ● Telephone 36-1-170439 ● Telex 226812